Active Galactic Nuclei [20 marks]

Active Galactic Nuclei (AGN) refer to energetic phenomena in central regions of some galaxies not directly attributable to stars. The two largest subclasses of AGNs are Seyfert galaxies and quasars, differing primarily in the amount of radiation emitted by the compact central source. In a typical Seyfert galaxy, the central source emits roughly the same amount of radiation as all its stars (i.e., ~ 10^{11} $L_{\odot}$), while in a quasar it is nearly a hundredfold more.

NGC5548 is one of the most extensively studied Seyfert in literature and today, we will make use of NGC5548 to determine some of the key features of an AGN.

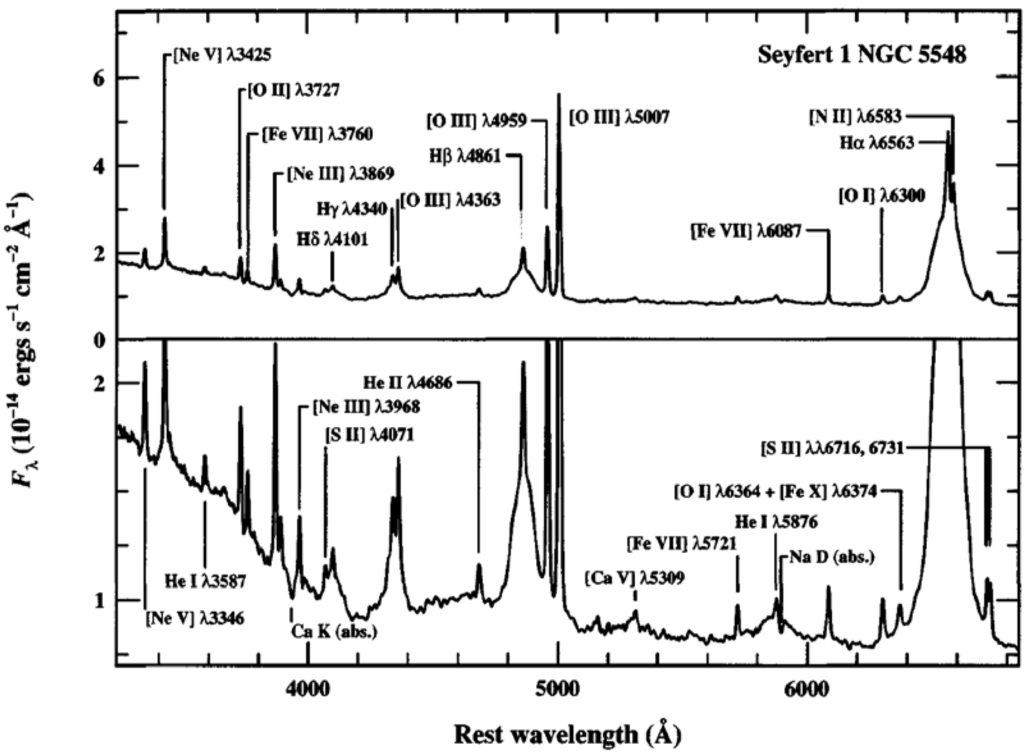


Figure 1 *The optical spectrum of the Seyfert 1 galaxy NGC 5548. Prominent broad and narrow emission lines are labelled, as are strong absorption features of the host galaxy spectrum. The vertical scale is expanded in the lower panel to show the weaker features. Note that the transition lines wavelength values provided are measured at rest.*

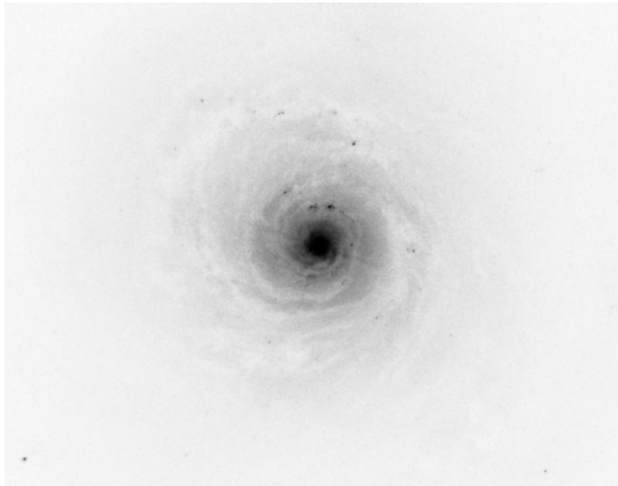


Figure 2 *NGC 5548 photographed by the Hubble Space Telescope. (Image inverted in black and white.) Credit: ESA/Hubble and NASA. (Acknowledgement: Davide de Martin)*

<u>Part I [15 marks]</u>

NGC 5548 is a spiral galaxy that is found in the constellation Bootes. It is also notably first identified in 1943 by Carl Seyfert to possess an active galactic nucleus, a Seyfert 1 galaxy. It is found that the radial velocity required for full width at half maximum (FWHM) of the broad components is about 5900 km s^{-1}, and the width of the narrow components is about 400 km s^{-1}. It is hence believe that the components originate from different regions of the Seyfert galaxy and are simply named 'broad' and 'narrow' regions. Their radial velocities however do give clues as to which physical region they originate.

Let's attempt to calculate all that is to be known from a spectral graph of such an object.

i) Which "region(s)" of the Seyfert galaxy do the Hβ transition line and the O III **primary transition line** lines originate from? Hence or otherwise, compute the upper and lower bound of the FWHM and the width lengths using the appropriate radial velocities. [3 marks]

For Hβ, it has a rest wavelength of 4861×10^{-10} m; O III with rest wavelength of 5007×10^{-10}m. By inspection of the spectra, Hβ originates from the broad region, while O III originates from the narrow region.
Thus, using the relativistic Doppler shift for $\pm v$, we would get 2 values of $\lambda_{observed}$ for the upper and lower bounds respectively for the two transitions lines each.

Thus we have 4958, 4766; 5014, 5000 angstroms respectively.

ii) Given that the observed spectral wavelength of the O III primary transition line is 5089 angstroms, calculate the redshift for NGC 5548 and hence its distance away from earth. State any assumptions used in your calculations. [3 marks]

$$\lambda_{\text{obs}} = (z+1)\lambda_{\text{emitted}}$$

Solving for the redshift, z

$$5089\times10^{-10} = (z+1)5007\times10^{-10}$$
$$z = 0.01638$$

Then since z is small, v/c is much smaller than 1 (Assumption 1)

$$z = \frac{v}{c}; v = H_0 d$$

We would find

$$d = 72 \text{ Mpc}$$

iii) It is known that NGC 5548's core is so luminous such that looking through a telescope it appears to be a star, hence 'quasi-stellar', albeit an unresolved star. By calculating the maximum resolution of the NGC 5548 by the 2.4m Hubble Space Telescope, observed in the visible range, suggest an appropriate upper bound to the diameter of the galactic nucleus in pc. Recall that the visible spectrum lies between 350nm to 700nm. Explain your answer. [2 marks]

From Rayleigh's Criterion

$$\theta = 1.22\frac{\lambda}{D}$$

For $\lambda = 3500\times10^{-10}$ m (lowest visible wavelength for maximum resolution) with D = 3m,

$$\theta = 1.78\times10^{-7}\text{m.}$$

As such,

$$\theta = \frac{\text{diameter}}{\text{distance}} = 1.22\frac{\lambda}{D}$$

Where we have used, $\text{Tan}\theta \approx \text{Sin}\theta \approx \theta$

$$\text{Diameter} = 12.81\text{pc}$$

iv) Calculate the absolute magnitude of NGC 5548 given that it has an apparent magnitude of 13.3. [1 mark]

$$13.3 - \text{M} = 5\text{Log}10[\frac{72 * 10^6}{10}]$$
$$\text{M} = -21.0$$

v) Given that NGC 5548 looks as shown above in Figure 2, suggest if the Tully Fisher relation would be useful in determining the luminosity of the object. [1 mark]

No, it would not be particularly useful given that the Tully Fisher relation works best for edge on spiral galaxies. In our case, NGC5548 is largely face-on and thus this would introduce a large amount of error in our calculations.

vi) It is understood that the broad emission lines observed in the spectrum of AGNs are formed in a region within/near the galactic nucleus, while that of narrow emission lines lie beyond galactic nucleus. Using Kepler's Laws, explain how can broad and narrow emission lines coexist for the same object (NGC 5548). [2 marks]

From K3L we have $T^2 \propto R^3$, which suggests also that $\frac{1}{R} \propto v^2$. Hence, at a distance further away, the orbital/radial velocity is expected to be smaller. Hence for the narrow emission lines which are further away from the nucleus experiences a smaller orbital/radial velocity while the broad emission lines exists for transitions within/near the galactic nucleus.

vii) The culprit for the broadening of the spectral lines in NGC 5548 is due to the accretion of matter onto supermassive black holes. Hence or otherwise, estimate the mass of the supermassive blackhole responsible for this object. [1 mark]

$$\mathrm{M} = \frac{rv^2}{G};$$

Do not use Schwarschild radius equation since we are not finding that. Instead, we can guess the mass of the supermassive black hole from the radial velocity of the components as shown from the spectrum. The above formula may be found from equating centripetal force with the gravitational force as it should be. We may then assume to use the higher radial velocity components to be closest to the accretion disk of the black hole:

$$M = \frac{\frac{12.81}{2} \times 9.4605284 \times 10^{15} m \times (5900 \times 10^3)^2 m^2 s^{-2}}{6.67384 * 10^{-11} kg^{-1} m^3 s^{-2}}$$

$$M = 1.031 \times 10^{41} \mathrm{kg} = 5.15 \times 10^{10} \mathrm{M}_\odot$$

viii) Suggest why using the smaller orbital velocity value (found for the narrow emission lines) will lead to a poorer estimate, even if all other necessary quantities are known. [1 mark]

Given that the narrow emission lines are found at a distance much further away from the nucleus, using the smaller velocity leads to us estimating a total mass than includes the torus and other materials found in the galaxy (even if the radius of the narrow region is known)

ix) From literature, the mass of the supermassive blackhole is approximately $M = 6.5 \times 10^7 \mathrm{M}_\odot$ Compare the literature value against the value you found in (viii). [1 mark]

The value found is much larger; the reason being we assumed the broadband region is located at the same distance as the resolution limit of the quasar. It is likely that the broadband region is located much closer in.

Part II [5 marks]

It is known that the torus of the AGN is formed at a distance r_d pc away from the core due to evaporation and it is proportional to the absolute luminosity, L (Barvainis, R. 1987, ApJ, 320, 537) of the nucleus. Thus, it is expected that there would be a time delay Δt days as the incident wavelength is re-emitted by the torus, which is simply the time taken for the photons emitted by the nucleus to reach the torus at a distance r_d pc away.

x) Assume that the region between the nucleus and the torus is fully cleared of materials (i.e. it is a vacuum) and that the time delay $\Delta t = 49.5$ days, what does this tell us about the nucleus radii and the mass of the supermassive blackhole? [2 marks]

The region cleared of material, r_d pc would extend all the way to $1.282\times10^{15}\mathrm{m} = \ 0.0416\ \mathrm{pc}$ for the give time delay. This tells us that our estimate of the nucleus radii in (iii) is indeed vastly wrong. This value would however give us a much closer estimate of the mass of the supermassive blackhole at $M = 3.34\times10^{8}\mathrm{M_\odot}$.

xi) Following which, we may check for mass loss by calculation of what is known as the Eddington limit, for beyond this limit, mass loss occurs for a spherically symmetric object in equilibrium. Hence or otherwise, use supporting calculations to determine if there is any possibility that the black hole region is losing mass. You are given that the Eddington Limit is $L_{ED} \cong 1.5\times10^{31}\frac{M}{M_\odot}W.$ for the supermassive blackhole. [3 marks]

We know that it's absolute bolometric magnitude is -21.0. From which can calculate its luminosity easily being $2\times10^{10}L_\odot = 7.71\times10^{36}$ W using the Absolute Bolometric Magnitude formula found in the formula booklet. For a mass of $6.5\times10^{7}\mathrm{M_\odot} = 1.3\times10^{38}\mathrm{kg}$, we would find the L_{ED} to be $9.75\times10^{38}\mathrm{W}$, so it seems that there's no mass loss in play, even in the worst case scenario where the luminosity of the galaxy is concentrated at the core.

An Astronomical Mystery: A Case of Identity [20 marks]

An astrophotographer has taken the following image over the course of a night.

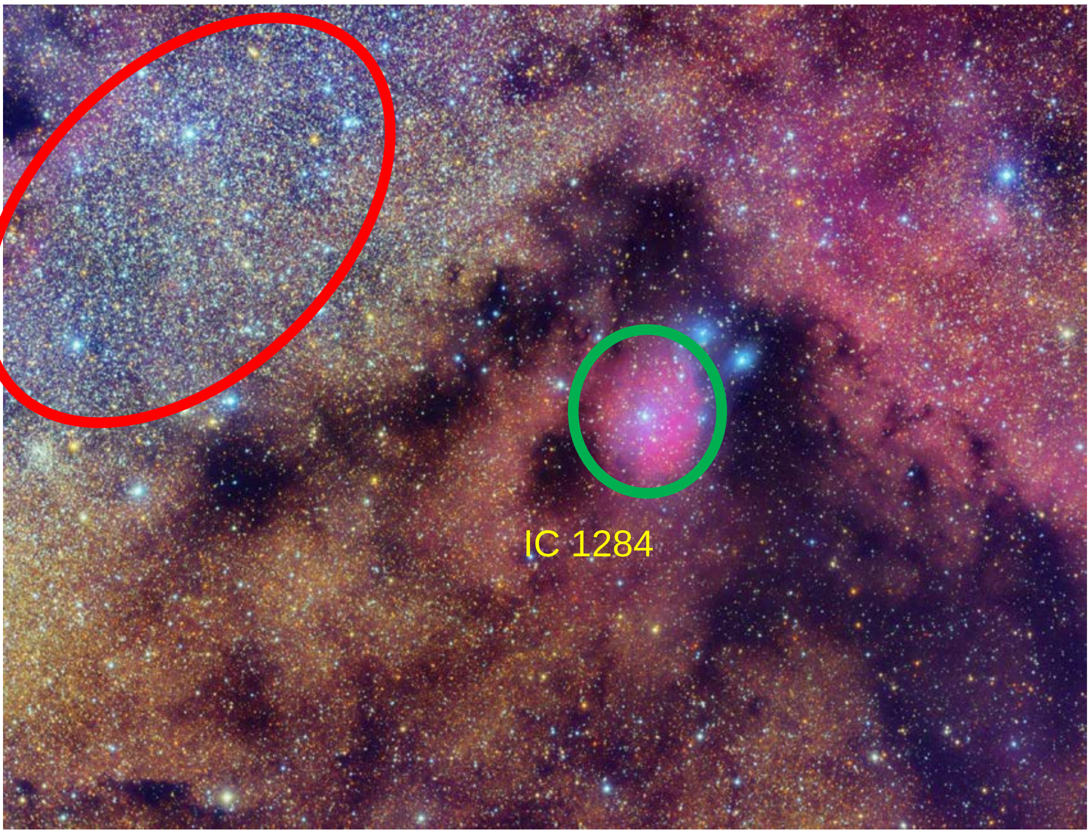


Figure 1 *IC 1284 and surrounding regions (Photograph by Ivan Bok)*

Much is going on in this image; there are at least two key objects of interest here, as circled. Let us first focus on IC 1284 (the circle). We seem to have a bright star, surrounded by a spherical halo of glowing reddish pink gas.

Part I: The Star [5 marks]

The central star of IC 1284 (HD 167815) is poorly studied in the existing literature. Here are some key data from several general surveys, as obtained from the SIMBAD database.

Parallax	2.80 milliarcseconds
Spectral Classification	B1.5III
Apparent magnitude (B)	7.70
Apparent magnitude (V)	7.61
Apparent magnitude (K, near-infrared)	6.957

i) Just by looking at its spectrum, we know that HD 167815 is a B-type giant star with a surface temperature of around 20,000K-25,000K. How do astronomers quickly differentiate a B-type star (from say, a K-type star), solely by looking at a spectrum?

[1 mark]

ii) Similarly, how do we infer a star's luminosity class, solely by looking at a spectrum (in other words, differentiate a giant star from a dwarf star)? [1 mark]
NB: We are only looking for a general list of feature(s), not a lengthy explanation of why these feature(s) work to distinguish these stars.

The presence/absence of specific absorption lines (e.g. ionized helium) helps us to classify stars under the standard Harvard Classification System (OBAFGKM). We can assign stars a luminosity class by examining the **width** of certain absorption lines.

A star with the same stellar classification as HD 167815 is α Lupi, the brightest star in Lupus. The table below summarizes some key information about this star

Parallax	7.02 milliarcseconds
Luminosity	25,000 $L_\odot$
Absolute magnitude (V)	-4.3
Apparent magnitude (B)	2.126
Apparent magnitude (V)	2.286
Apparent magnitude (K, near-infrared)	2.668

iii) Due to intervening dust, α Lupi is visually dimmed by an extinction factor, A(V). A(V) is expressed in terms of magnitudes, and is related to the apparent and absolute magnitude by the following equation:

$$m(V) = M(V) + 5\log_{10} d - 5 + A(V)$$

Where d is in parsecs as usual.

Given the similar spectral classes of α Lupi and HD 167815, we may assume that both stars have similar luminosities and absolute magnitudes. Further, we know that due to interstellar extinction, only a certain fraction of a star's total light reaches us. Hence, compute the amount of light that reaches us from both stars. Express your answer as a percentage of the original unextincted brightness. [3 marks]

Compute distance to HD 167815

$$d = \frac{1}{2.8\times10^{-3}} \approx 357.14\,pc$$

Dimming for α Lupi and HD 167815 respectively:

$$2.286 = -4.3 + 5\log 142.45 - 5 + A(V)$$
$$A(V) = 2.286 + 4.3 - 5\log 142.45 + 5 = 0.84\,mag$$

$$7.61 = -4.3 + 5\log 357.14 - 5 + A(V)$$
$$A(V) = 7.61 + 4.3 - 5\log 357.14 + 5 \approx 4.15\,mag$$

Simply repurpose the relationship between luminosity and absolute magnitude. You should obtain upon inspection

$$\frac{L_{final}}{L_{initial}} = 10^{\frac{-A(V)}{2.5}}$$

This yields 46.1% and 2.2% for α Lupi and HD 167815 respectively.

Part II: The Nebula [9 marks]

How about IC 1284 itself? Unlike far more famous nebulae, IC 1284 is almost perfectly spherical. This suggests that IC 1284 is a classic example of a special type of nebulae, whose formation process was first described 80 years ago.

Let us start with a star inside a cloud of hydrogen gas of uniform density. The star emits high-energy photons, which completely ionises the hydrogen in a shell around the star. Some of the free protons and electrons then recombine and emit photons (see equation below), generating the light that illuminates the nebula.

$$p^{+} + e^{-} \rightarrow H + \gamma$$

Under ideal conditions, the result is what we see right now: a glowing sphere of ionized hydrogen surrounding a star.

Let us consider a $1\,m^3$ unit volume of the resultant nebula, containing a number of free electrons and protons (n_e and n_p) respectively. The rate of net recombination in that unit volume is then given by the rate equation:

$$Rate = \alpha(T) n_e n_p$$

$a(T)$ represents the net recombination coefficient, equalling $2.6\times10^{-19}\,m^{-3}\,s^{-1}$ at a temperature of 10,000K. (If this *looks* like a typical 2nd order rate equation in JC Chemistry Kinetics, that's because it is.)

iv) How does the rate of recombination change with temperature? In other words, does recombination occur faster or slower at higher temperatures? Why? [1.5 marks]
Hint: consider the physical mechanism behind recombination.

Unlike most chemical reactions, the rate of recombination **slows** down at higher temperatures. This is because it is easier for low-energy ions to recombine: put simply, free protons find it harder to "grab" onto fast moving electrons.

v) An astronomer boldly claims that $n_p = n_e$ for all regions of the nebula. Is this a justified assumption? Briefly explain. [0.5 marks]

Yes, it is a good assumption: the initial gas must be electrically neutral.

vi) Suppose that HD 167815 supplies Q ionising photons per second to IC 1284. Hence or otherwise, derive an expression for the equilibrium radius of the nebula, R_S. Assume perfect absorption of ionising photons. [2 marks]

At equilibrium, the total rate of net recombination equals the total rate of ionization

$$\text{Total net recombination rate} = \frac{4}{3}\pi {R_S}^3 \alpha(T) {n_e}^2$$

$$Q = \frac{4}{3}\pi (R_S)^3 \alpha(T) {n_e}^2$$

$$R_S = \sqrt[3]{\frac{3Q}{4\pi\alpha(T){n_e}^2}}$$

vii) Given that, the imaging setup used has a pixel scale of 2.48 arcseconds/pixel and that IC 1284 spans a diameter of 370 pixels at its narrowest, calculate the approximate radius of IC 1284 in light years. [2 marks]

$$\text{Radius (in arcseconds)} = 2.48''\times 185 = 458.8''$$

$$\tan(458.8'') = \frac{R_S}{357.14\ pc}$$

$$R_S = 0.794\ pc = 2.59\ ly$$

viii) A B1.5-type blue giant like HD 167815 produces around 4 $\times 10^{47}$ ionising photons per second. Hence or otherwise, find the average number of protons/electrons per unit volume in the nebula surrounding IC 1284. Assume equilibrium. [1 mark]

Plugging in the values, we get:

$$4 \times 10^{47} = \frac{4}{3}\pi(0.794\ pc)^3 (2.6\times 10^{-19}\ m^{-3}) {n_e}^2$$

$$n_e \approx 158.02 \times 10^8$$

In other words, the molecular cloud surrounding HD 167815 has an average density of around 158 protons/electrons per cubic centimetre

ix) Given that the Hα line has a wavelength of 656.28 nm and around 40% of all recombining hydrogen gas emits Hα, what is the luminosity of IC 1284 in Hα? Express your answer in terms of solar luminosities. [2 marks]

Since the rate of net recombination = rate of ionisation

$$\begin{aligned}\text{H}\alpha\text{ emission rate } &= 0.4 * Q \\ &= 1.6\times 10^{47}\ photons\ s^{-1}\end{aligned}$$

The energy of a photon is E=hf, which gives us:

$$L_{\text{H}\alpha} = (1.6\times 10^{47}) * \frac{hc}{656.28\ nm} = 4.843\times 10^{28}\ W \approx 126\text{L}_\odot$$

Part III: The Big Picture [6 marks]

Now we return to the oval. This oval is large (spanning nearly 2 degrees across), and boasts of several curious properties. What is its true nature?

x) An astronomer plots out a color-magnitude diagram for surveyed stars in the oval brighter than an apparent magnitude of 14 (V). See the figure below. The resultant color-magnitude diagram has extremely high scatter and does not display an obvious main sequence. Given that the data is complete and accurate, suggest the most important reason for this [2 marks]

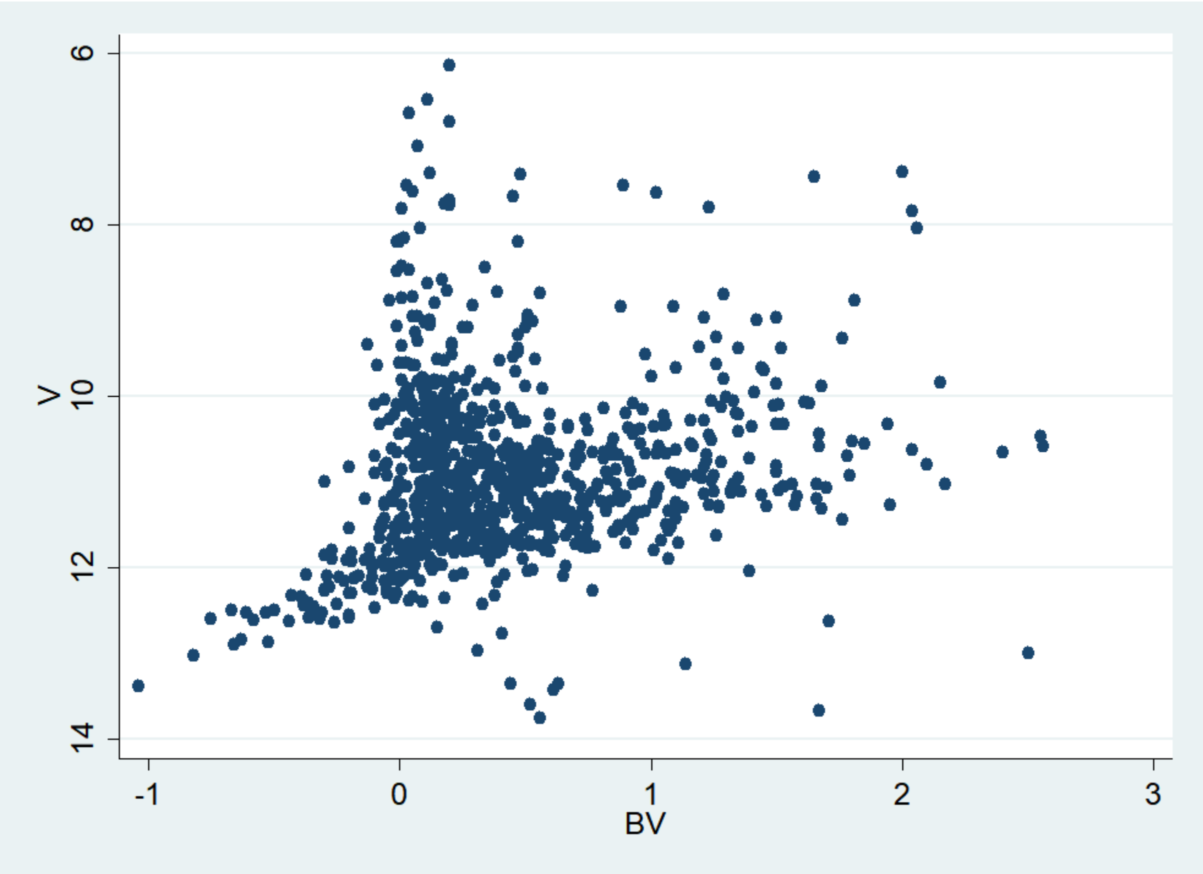


Figure 2: *CMD for a sample of stars within 1 degree from the object centre, brighter than mag +14 (V). Note that the X axis measures B-V.*

The stars are not physically close to each other in space and/or are not born at the same time. Thus, the view contains a jumble of evolved and nearby stars at varying distances, while main sequence stars of similar colour end up with wildly different apparent magnitudes.

xi) Refer to Figure 1. Observe that other than the stars in the oval, most of the stars appear highly yellow. Explain why these stars are so yellow. [1 mark]

Interstellar dust and gas preferentially absorbs blue light, leaving only the redder wavelengths. This is the phenomenon known as interstellar reddening [1 mark].

In fact, we can see this in HD 167815: even though it is a blue giant, the intervening dust causes it to appear brighter in red than in blue light.

xii) In the 1800s, this oval was described as “a large nebulosity containing many stars”. With better optics, we know that this is false: this oval is not a nebula. Hence or otherwise, discern the true nature of the object in this oval. Your answer should reference specific evidence in the images and/or answers in the previous questions. [3 marks]

From the CMD: we know the stars in the oval are not physically associated in space. From the photos/A(V) calculations: we know this oval is located near the galactic core and is surrounded by large amounts of gas and dust. Close examination of Figure 1 also reveals that unlike most clusters, there’s no well-defined stellar boundary for the oval. Rather, the boundary is set by the gas/dust clouds surrounding the “oval”, as well as the resultant colour/brightness changes. All these lines of evidence lead to one unmistakable conclusion: **the oval does not contain a real “object”**. Rather, it is merely a region of the central Milky Way that is relatively free of gas and dust. Thus, when we peer through this window, we are looking through the spiral arms and witnessing all the stars in between, without significant hindrance by dust.

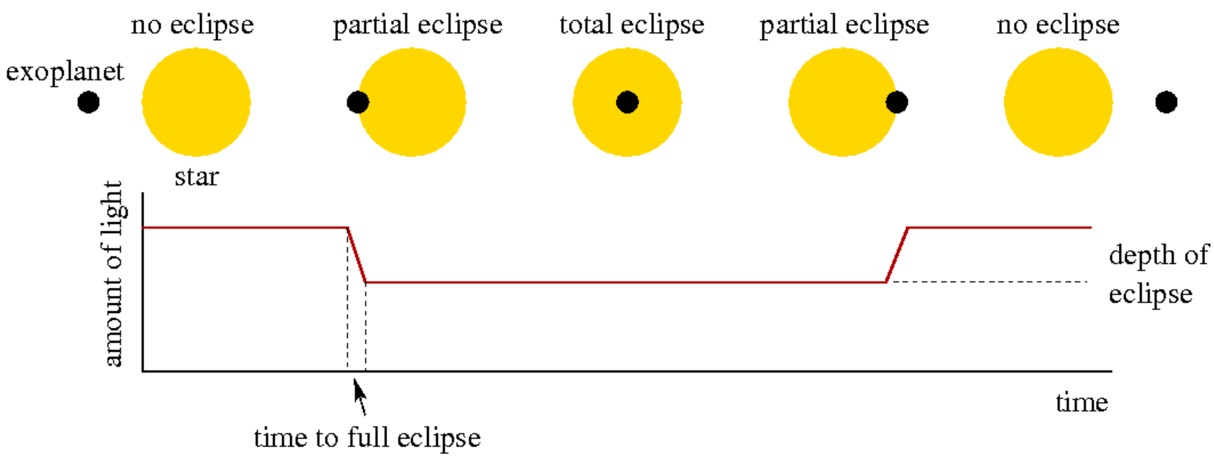
[1 mark for correct identification of object nature, 1 mark for each correct interpretation of evidence, capped at 2 marks]

Incidentally, the oval in question contains M24, the Sagittarius Star Cloud

Figure 3 *A wide field shot of the Milky Way. Oval drawn to approximately the same scale. Taken by Kia Yee*

Finding exoplanets… IT'S A TRAP$_{\text{PIST1}}$ (22 marks)

Exoplanets can be detected once they transit a host star, causing a dip in the host star's light curve periodically (Image modified from galileospendulum.org):



Let the stellar mass be M_*, stellar radius be R_*, planet radius be R_p, the orbital semi-major axis be a, orbital inclination be i and orbital period be P.

The transit depth can be expressed as ΔF, or change in flux (due to the planet blocking the host star). Let F_n be normal flux without a transit, and F_t be transit flux observed:

$$\Delta F = \frac{F_n - F_t}{F_n} = \left(\frac{R_p}{R_*}\right)^2 \cdots\cdots\cdots \textit{Equation } 1$$

Since the transit curve can effectively be expressed as a periodic sine function, it can be expressed as a ratio of duration of the flat part of transit (t_F) to the total transit duration (t_T):

$$\frac{\sin\left(\frac{t_F \pi}{P}\right)}{\sin\left(\frac{t_T \pi}{P}\right)} = \frac{\sqrt{\left(1 - \frac{R_p}{R_*}\right)^2 - \left(\frac{a}{R_*}\cos i\right)^2}}{\sqrt{\left(1 + \frac{R_p}{R_*}\right)^2 - \left(\frac{a}{R_*}\cos i\right)^2}} \cdots\cdots\cdots \textit{Equation } 2$$

Next, the total transit duration, t_T can be expressed as:

$$t_T = \frac{P}{\pi}\arcsin\left(\frac{R_*}{a}\sqrt{\frac{\left(1 + \frac{R_p}{R_*}\right)^2 - \left(\frac{a}{R_*}\cos i\right)^2}{1 - \cos^2 i}}\right) \cdots\cdots\cdots \textit{Equation } 3$$

To visualise all of the above, this diagram was drawn by Seage & Mallen-Ornelas, 2003:

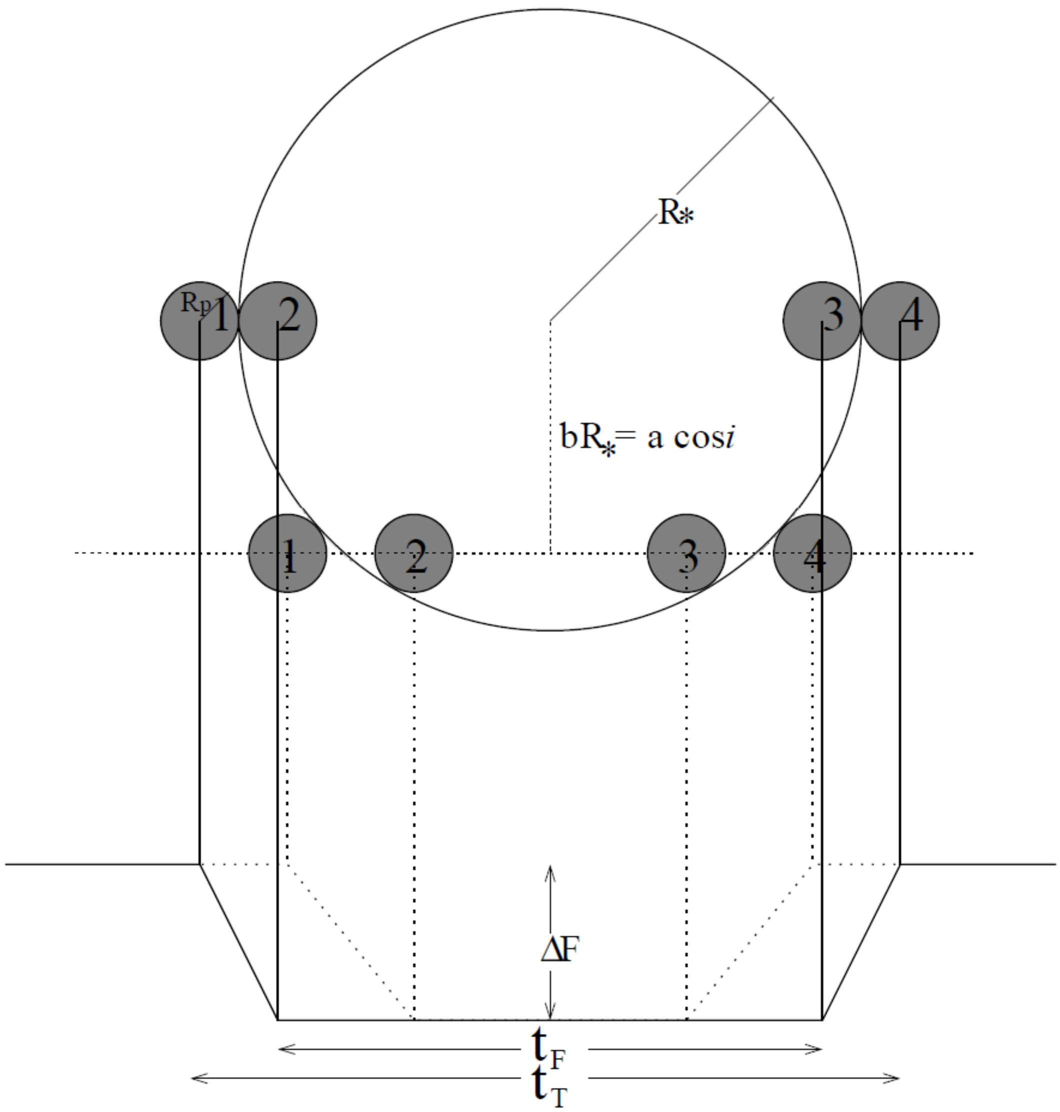


Fig. 1. Definition of transit light-curve observables. Two schematic light curves are shown on the bottom (solid and dotted lines), and the corresponding geometry of the star and planet is shown on the top. Indicated on the solid light curve are the transit depth ΔF, the total transit duration t_T, and the transit duration between ingress and egress t_F (i.e., the ‘‘ flat part ’’ of the transit light curve when the planet is fully superimposed on the parent star). First, second, third, and fourth contacts are noted for a planet moving from left to right. Also defined are R_*, R_p, and impact parameter *b* corresponding to orbital inclination *i*. Different impact parameters *b* (or different *i*) will result in different transit shapes, as shown by the transits corresponding to the solid and dotted lines.

Seager, S., & Mallen-Ornelas, G. (2003). A unique solution of planet and star parameters from an extrasolar planet transit light curve. *The Astrophysical Journal*, *585*(2), 1038.

A) Using the above equations and Fig. 1, show that semi-major axis, a, can be expressed as: [3]

$$a = R_* \sqrt{\frac{\left(1+\sqrt{\Delta F}\right)^2 - b^2\left[1-\sin^2\left(\frac{t_T \pi}{P}\right)\right]}{\sin^2\left(\frac{t_T \pi}{P}\right)}} \cdots\cdots\cdots \; Equation\ 4$$

$$Since\ t_T = \frac{P}{\pi}\arcsin\left(\frac{R_*}{a}\sqrt{\frac{\left(1+\frac{R_p}{R_*}\right)^2 - \left(\frac{a}{R_*}\cos i\right)^2}{1-\cos^2 i}}\right) \;\&\; bR_* = a\cos i,$$

$$\cos i = \frac{bR_*}{a};\, sub\ into\ equation:\ t_T = \frac{P}{\pi}\arcsin\left(\frac{R_*}{a}\sqrt{\frac{\left(1+\frac{R_p}{R_*}\right)^2 - \left(\frac{a}{R_*}\frac{bR_*}{a}\right)^2}{1-\left(\frac{bR_*}{a}\right)^2}}\right)$$

$$\Delta F = \frac{F_n - F_t}{F_n} = \left(\frac{R_p}{R_*}\right)^2;\, sub\ into\ equation:\ t_T = \frac{P}{\pi}\arcsin\left(\frac{R_*}{a}\sqrt{\frac{(1+\Delta F)^2 - b^2}{1-\left(\frac{bR_*}{a}\right)^2}}\right)$$

$$Rearrange\ both\ sides: \sin\left(\frac{t_T \pi}{P}\right) = \frac{R_*}{a}\sqrt{\frac{(1+\Delta F)^2 - b^2}{1-\left(\frac{bR_*}{a}\right)^2}}$$

$$\text{Cross multiply over: } a\sqrt{1-\left(\frac{bR_*}{a}\right)^2} = R_*\sqrt{\frac{(1+\Delta F)^2 - b^2}{\sin^2\left(\frac{t_T \pi}{P}\right)}}$$

$$make\ a\ the\ subject: a\sqrt{1-\left(\frac{bR_*}{a}\right)^2} = \sqrt{a^2-(bR_*)^2};\, square\ both\ sides, move\ (bR_*)^2 over$$

$$a^2 = {R_*}^2\frac{(1+\Delta F)^2 - b^2}{\sin^2\left(\frac{t_T \pi}{P}\right)} - b^2{R_*}^2$$

$a^2 = {R_*}^2\frac{(1+\Delta F)^2-b^2}{\sin^2(\frac{t_T\pi}{P})} - \frac{b^2{R_*}^2(\sin^2(\frac{t_T\pi}{P}))}{\sin^2(\frac{t_T\pi}{P})}$; factor out b^2 and square root both sides. See Eqn 4.

WARNING: TRAP QUESTION. NOT FOR THE FAINT OF HEART.

B) Show that, when orbital inclination is assumed to be 90°, i.e. the planet's orbit is exactly on the plane of the host star's ecliptic, the following equations can be obtained: [2]

$$t_T = \frac{P}{\pi}\arcsin\left(\frac{R_*}{a}\left(1+\frac{R_p}{R_*}\right)\right) \cdots\cdots\cdots \textit{Equation } 5$$

$$a = R_* \frac{\left(1+\sqrt{\Delta F}\right)}{\sin\left(\frac{t_T\pi}{P}\right)} \cdots\cdots\cdots \textit{Equation } 6$$

Sub in cos (π/2 OR 90°) = 0, then show full workings. SHOULD BE EASY!

$$t_T = \frac{P}{\pi}\arcsin\left(\frac{R_*}{a}\sqrt{\frac{\left(1+\frac{R_p}{R_*}\right)^2-\left(\frac{a}{R_*}\cos i\right)^2}{1-\cos^2 i}}\right) \quad a = R_*\sqrt{\frac{\left(1+\sqrt{\Delta F}\right)^2-b^2\left[1-\sin^2\left(\frac{t_T\pi}{P}\right)\right]}{\sin^2\left(\frac{t_T\pi}{P}\right)}}$$ (no highlighted terms)

Just as you're about to hit the **PANIC button**, you're in luck: Experts have already helped you to derive stellar density, ρ_*, which can be expressed as:

$$\rho_* = \left(\frac{M_*}{R_*{}^3}\right) = \left(\frac{4\pi^2}{P^2G}\right)\left(\frac{\left(1+\sqrt{\Delta F}\right)^2-b^2\left[1-\sin^2\left(\frac{t_T\pi}{P}\right)\right]}{\sin^2\left(\frac{t_T\pi}{P}\right)}\right)^{\frac{3}{2}} \cdots\cdots\cdots \textit{Equation } 7$$

C) What is the key assumption made, in order to obtain equation 7 from equation 4 in the process? (Hint: You can derive equation 7 from equation 4 in case you cannot figure it out, and you will notice a certain term is missing. Actual derivation workings NOT required.) [1]

The mass of the planet was assumed to be insignificant compared to the mass of the star; would not affect the value of density much.

D) Briefly state two other key assumptions or conditions that need to be made or currently exists, in order to successfully use the above equations in order to compute parameters (you do not need to explain why). [1]

Any of the following: The planet orbit is approximated as (almost perfectly) circular; The companion planet is dark/ does not give out any light; Companion planet is spherical; Companion planet does not possess a large moon of significant size; The stellar mass-radius relation is known; The light comes from a single star, rather than from two or more blended stars/ sources; The eclipses have flat bottoms, i.e. companion is fully superimposed on the central star's disk; The period can be derived entirely from the light curve; and any other reasonable assumptions.

If you're wondering if this question was indeed a trap… you're probably right. Just as you thought things couldn't get any worse, you're presented with a case study of the planetary system, TRAPPIST-1 (Disclaimer: this exists in real life, discovered by the Transiting Planets and Planetesimals Small Telescope in Belgium. Belgium has a famous beer brand named – you guessed it – Trappist Beer).

Table 1: Data pertaining to host star, TRAPPIST-1a

Spectral type	M8V
Distance	39.5 ly (12.1 pc)
Absolute magnitude (MV)	18.4
Mass	0.08 $M_\odot$
Radius	0.114 $R_\odot$
Luminosity (visual, LV)	0.00000373 $L_\odot$
Surface Temperature	2550±55 K
Metallicity	0.04
Rotation	3.30 days
Age	3–8 Gyr

Table 2: Data about the TRAPPIST-1 planetary system, with some information missing.

Designation of planet	Mass ($M_\oplus$)	Semimajor axis (AU)	Orbital period (days)	Eccentricity	Inclination (°)	Radius ($R_\oplus$)
b	0.79 ± 0.27	0.01111	1.51	0.019	89.65	1.086
c	1.63 ± 0.63	0.01522	2.42	0.014	89.67	1.056
d	0.33 ± 0.15	0.021	4.05	0.003	89.75	0.772
e	0.56 ± 0.24	0.028	6.10	0.007	89.86	0.918
f	0.36 ± 0.12	0.037	9.21	0.011	89.68	1.045
g	0.57 ± 0.04	0.045	12.35	0.003	89.71	1.127
h	0.086 ± 0.08	0.060	18.76	0.086	89.8	0.715

~~Eeeeek) Using the most complicated formula you've seen, calculate the transit duration of…~~

Just as you're about to hit the **PANIC button** again, your Astrochallenge intellect informs you that Equations 3 and 4, while accurate and important for other planets, aren't necessary for the TRAPPIST-1 star system. The simpler Equations 5 and 6 would suffice.

E) With this in mind, answer the following questions instead:

(i) Why can you use Equation 5 and 6 for the TRAPPIST-1 system? [0.5]

Orbital Inclination is almost 90°, and thus would not affect calculations much.

(ii) Given that the transit duration of TRAPPIST-1b is approximately 0.025 days, calculate the semi-major axis of TRAPPIST-1b, leaving your answer in **AUs**. [1]

Equation 6: (Rstar*(1 + (1.086*RE)/Rstar)/Sin[((0.025)*\[Pi])/1.51])/AU = 0.0110867 AUs (close enough to actual value, 0.01111)

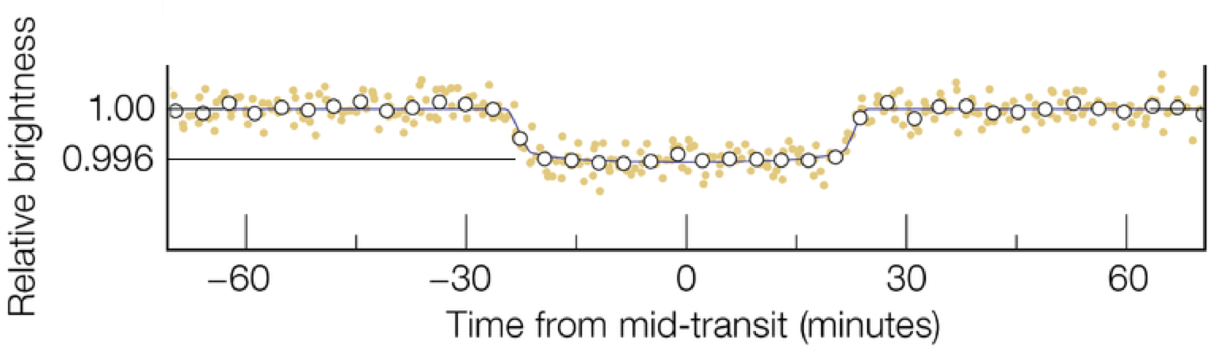
(iii) Calculate the transit duration of TRAPPIST-1d, leaving you answer in number of **days**. [1]

Equation 5: (4.05/\[Pi])*ArcSin[(Rstar/(0.021*AU))*(1 + (0.772*RE/Rstar))] = 0.0345672 days (approximately 50 minutes; see part F) for more details why this is important)

(iv) Given that the transit duration of TRAPPIST-1e is approximately 0.0395 days, calculate the radius of TRAPPIST-1e, leaving your answer as a ratio compared to Earth's Radius (**$R_\oplus$**). [1.5]

Rewrite Equation 5: (((((0.028*AU)/Rstar)*Sin[(0.0395*\[Pi])/6.1]) - 1)*Rstar)/RE = 0.925679 Earth Radii (Actual: 0.918; close enough as well!)

The following is a light curve of one of the planets in TRAPPIST-1.



F) Determine which planet is featured by the above light curve, showing clear evidence why you think so. [2]

Many methods, but this can be reasoned out;

1. Convert 0.0345672 days to minutes, and you get approx. 50 min: which is the rough duration of transit! → planet TRAPPIST-1d is a prime suspect.

2. Delta F of planet 1d = 0.000384588, which is roughly 0.004, = ((0.772 RE)/(0.114 RS))^2. CONFIRMED.

3. TRAPPIST-1h is too far away and would have a way longer transit; I literally gave away the transit timings for 1b and 1e as well, and by now one should realise that shorter orbital periods give shorter transits; Accept quantitative common-sense answer!

Correct planet, 1 mark. Reasonable reason = 1 mark.

G) Tired of all the calculation questions? Here's some theory and a little controversy:

The inner and outer limits of the Habitable Zone of TRAPPIST-1a is determined to be a distance between 0.024 AU and 0.048 AU by one group of experts (Kopparapu et al., 2013, 2014), according to the following:

Inner limit: "runaway greenhouse limit where a planet's temperature would soar even with no CO2 present and lose all of its water in a geologically brief time in the process".

Outer limit: "maximum greenhouse limit of a CO2-rich atmosphere where the addition of any more of this greenhouse gas would not increase a planet's surface temperature any further".

(i) State the importance of the habitable zone in astrobiology and the implications of the above result pertaining to **all** the planets in the TRAPPIST-1 star system. [1.5]

Habitable zone: liquid water is a universal solvent for many of life's biochemical processes and would likely support extra-terrestrial life. [0.5 mark]

TRAPPIST-1b, c and d are below the inner limit and are unlikely to possess liquid water;

TRAPPIST-1e, f and g are within the habitable zone, thus containing liquid water;

TRAPPIST-1h is beyond the outer limit and would only have frozen water. [1 mark for all, 0.5 if rough idea stated without reference to planets]

Numbers for Habitable Zone: http://www.centauri-dreams.org/?p=37225

Other experts disagreed, arguing that due to the combined effect of tidal heating and tidal locking, up to all of the planets might be suitable for life or none of them might be suitable at all. Optimistic experts saw a loophole in the habitable zone's definitions which would allow life on these planets, while Pessimistic experts cited the Rare Earth Hypothesis and why these planets are all hostile for life. **You have to pick a side.**

(ii) **Briefly suggest** how tidal heating and tidal locking **EITHER improves OR hampers** the feasibility of life for the TRAPPIST-1 system. [1.5]

Optimists:

Tidal heating: Allows 1h to have sufficient thermal energy for liquid water

Tidal locking: Only one side of 1b, c, d will be 'baked' while the other side is frozen; life can still thrive on the terminator line

Pessimists:

Tidal heating: Highly active volcanoes make it difficult for life to develop

Tidal locking: Very strong winds will encircle the planet, and large temperature differences prevent the distribution of life

(Accept any other reasonable answers on both sides)

iii) **Suggest two** other methods other than the transit method for discovering exoplanets, and **briefly** state their principles. [2]

Radial Velocity/ Doppler spectroscopy: Stars will 'wobble' in orbit

Gravitational microlensing: Planet gravity can affect light

Direct imaging: If large and near enough to take a photo

And any other viable answers/ variants of the above

One final obstacle awaits you: A dreaded graphing question requiring everything you've seen so far and your wits. (Hint: It would be helpful to simplify Equation 2 for this following section. Workings are recommended but not strictly required for the final answer, though they can salvage your marks and give far more accurate diagrams. Feel free to use a pen/ highlighter of a different colour if it helps.)

H) Sketch the following, with both light curves on the same diagram, labelling each line as accurately and clearly as possible: [4]

(i) The transit curve of TRAPPIST-1c, assuming negligible inclination. Recommended units: Relative brightness/ flux compared to usual (1 $F_{TRAPPIST\text{-}1a}$) and time from mid-transit in minutes.

(ii) The transit curve of TRAPPIST-1c but with twice the current planet radius, with everything else remaining the same. (assume negligible inclination as well)

(i)

(2.42/\[Pi])*ArcSin[(Rstar/(0.01522*AU))*(1 + ((1.056*RE)/Rstar))] = 0.0291146 days = about 42 minutes

Delta F: ((1.056 RE)/(0.114 RS))^2 = 0.007; i.e. 0.993 of flux

$$\frac{\sin\left(\frac{t_F \pi}{P}\right)}{\sin\left(\frac{t_T \pi}{P}\right)} = \frac{\left(1 - \frac{R_p}{R_*}\right)}{\left(1 + \frac{R_p}{R_*}\right)} \ldots Equation\ 2\ modded$$

(If you know what you're doing, you don't need to calculate tF after you have tT; the ratio from the other side of this equation will tell you how much shorter the tF line is!)

(1 - ((1.056*RE)/Rstar))/(1 + ((1.056*RE)/Rstar)) = 0.844, i.e. tF is about 0.85 times between tT

(ii)

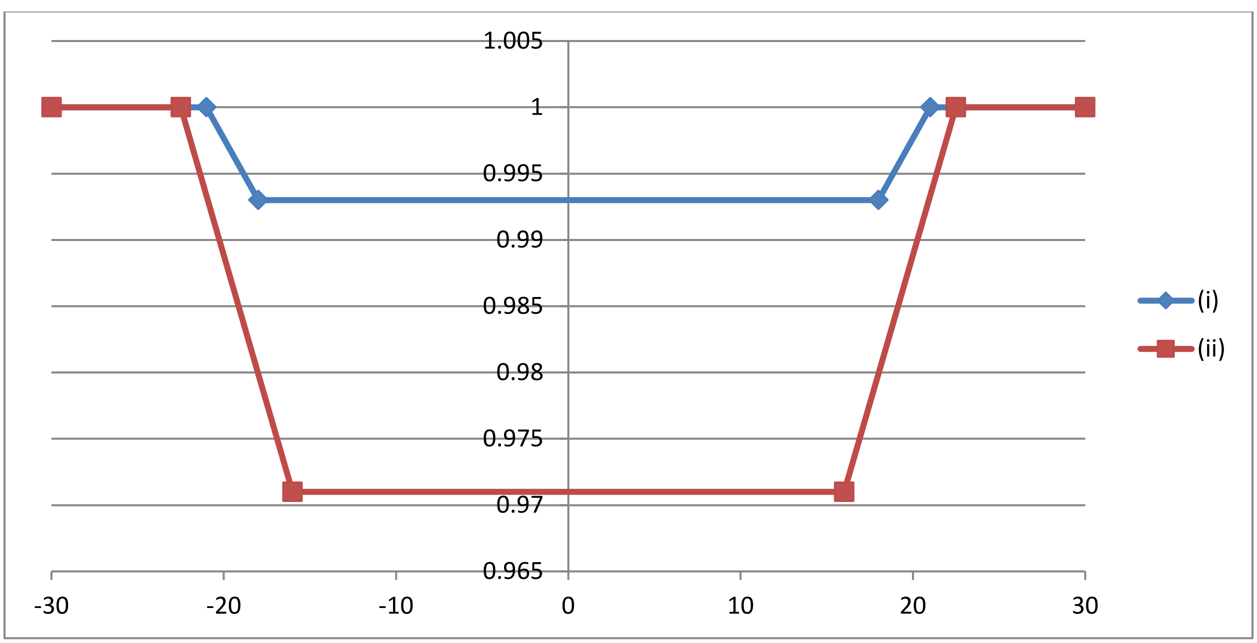
(2.42/\[Pi])*ArcSin[(Rstar/(0.01522*AU))*(1 + ((2.112*RE)/Rstar))] = 0.0313925 days = about 45 minutes, i.e. transit will last a bit longer if so!

Delta F: ((2.112 RE)/(0.114 RS))^2 = 0.029; i.e. 0.971 flux, or exactly 4 times as deep!

(If you're smart, you can immediately guess that delta F is 4 times from Eqn 1; no workings)

(1 - ((2.112*RE)/Rstar))/(1 + ((2.112*RE)/Rstar)) = 0.71, I.e. tF is about 0.7 times between tT

Graph:



Correct delta F ratios: 1 mark

Big planet dips first and emerges later by a little bit: 1 mark

Correct duration of total transit and full transits: 2 marks

(Note: graph is created with approximate numbers in Excel)

(marks can also be awarded or deducted on a case by case basis)

In case you're still trapped by TRAPPIST-1, press this button.
(Disclaimer: the DRQ will not explode)

[DON'T PANIC]*

***Don't throw in the towel. Read the question carefully, move on to an easier section, and/or ask for assistance from your team. You can do it!**

In a Coma over Coma [20 marks]

The Coma Star Cluster (Melotte 111) is one of the more prominent clusters in the northern spring sky. Located in Coma Berenices, the cluster center has approximate coordinates RA: 12h 25m and Dec: +25° 51′. It should not be confused with the Coma Cluster, a rich galaxy cluster that also lies in Coma Berenices. Using publicly available data, we've provided 2 copies of the Coma Star Cluster's colour-magnitude diagram (CMD), attached below.

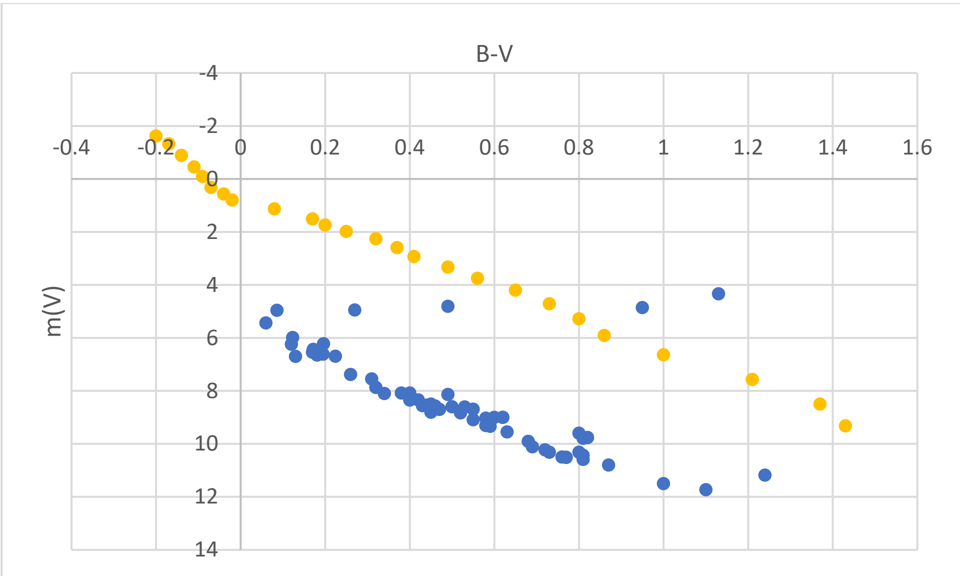
Part I: A quick and dirty estimate [8 marks]

i) An astronomer points out that interstellar gas between us and the Coma Star Cluster will absorb light from these stars, thus interfering with our measurements. This will thus cause the observed CMD to differ from the true CMD of the cluster, affecting the conclusions that we will draw. Are these concerns significant? Explain. [2 marks]

No. The Coma Star Cluster is located far from the galactic plane, and thus there is little intervening gas/dust to affect our observations. In fact, it is located near the North Galactic Pole!

Suppose that the CMDs provided have been corrected for the concerns raised in question (i). Appendix I contains data for the simulated Zero-Age Main Sequence (ZAMS) for stars between 0.1 to 7 solar masses. The ZAMS essentially marks where/when stars of each mass first hit the main-sequence.

ii) On the same axes, plot the ZAMS onto the **first** CMD provided [2 marks]. Hence or otherwise, determine the approximate distance to the Coma Star Cluster. Hint: like any good empiricist, you probably should make more than 1 measurement. [1 mark]



Acceptable range for distance modulus u= (5.15 ± 0.3)

$$d = 10^{\frac{5.15}{5}+1} \approx 107\ pc$$

iii) State the coordinates of the main-sequence turnoff point [1 mark]

From data, [0.06, 5.436] (error of 0.05/0.2 is acceptable)

iv) The main sequence lifetime of a star is approximately given by:

$$t_{ms} \propto \frac{M}{L}$$

Using the mass-luminosity relationship, derive an equation linking the main sequence lifetime of any star to the main sequence lifetime of the sun. [1 mark]

Hence or otherwise, estimate the age of the Coma Star Cluster. A ballpark estimate for the main-sequence lifetime of the Sun is 10 billion years. [1 mark]

Just plug in the mass-luminosity relationship: $L \propto M^{3.5} \rightarrow t_{ms} \propto M^{-2.5}$. To remove the constant of proportionality, divide throughout to get

$$\frac{t_{ms}}{t_{\odot}} = (\frac{M}{M_{\odot}})^{-2.5}$$

The star at the main-sequence turnoff has a B-V value of 0.06. The star with the closest B-V value to this on the ZAMS has an initial mass of 2.2 solar masses. Plug in the values to obtain 1.39 billion years.

Part II: Improving the estimate [5 marks]

As you might expect, we can improve on this crude estimate. One of the key concerns we might have over our ballpark estimates is the exact functional form of the mass-luminosity relationship. The mass-luminosity relationship is a general formula: there is strong reason to believe that the mass-luminosity relationship depends on metallicity and the exact stellar masses under consideration.

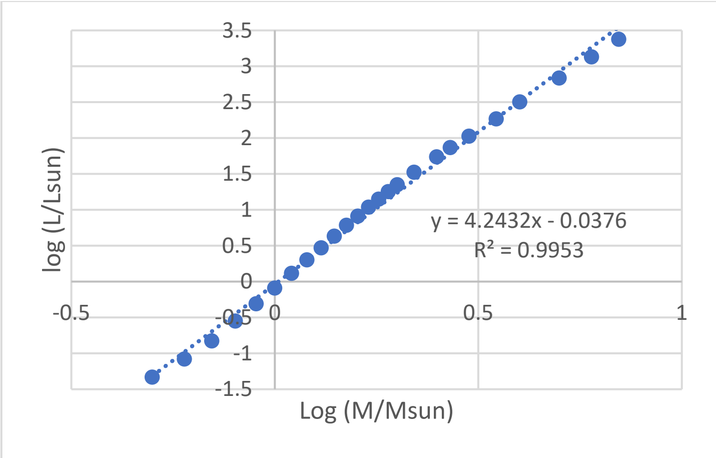
v) Linearise the mass-luminosity relationship [1 mark].

Let α denote the coefficient of the exponent to be estimated. Then

$$L = kM^{\alpha}$$
$$\log L = \log k + \alpha \log M$$

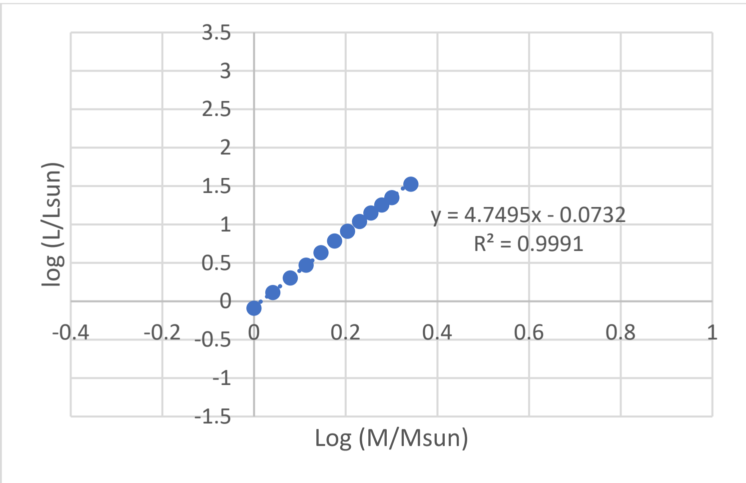
vi) By plotting suitable values on the blank graph provided, use the ZAMS data to estimate the exact form of the mass-luminosity relationship for values of interest. For your ease, we've plotted the point for a solar mass star. [3 marks]

Using solar units, the graph for the full sample is then:



As alluded to in the text above, note the slight bump between log(M/Msun) = 0 and log(M/Msun) = 0.5. Thus, we will obtain a better estimate if we confine our plotting to the mass range of interest (1-2.2 solar masses). [bonus 1 mark if students realise this]. Accept gradients within a 0.2 interval.

The restricted sample gives us:



vii) Hence or otherwise, revise your age estimate in part 4. [1 mark]

With the revised estimate for α, we get 775M and 520M for the full and restricted sample respectively. Note: the age derived from the restricted sample neatly concurs with the simple isochrone fit in the last section.

Part III: The actual work [7 marks]

In practice, astronomers determine the age of star clusters by using the concept of stellar isochrones. A stellar isochrone is a curve connecting stars of equal age on a HR diagram. By fitting these curves onto a cluster CMD, we can obtain a much more accurate estimate of a cluster's true age. For this express purpose, we have provided a set of 9 stellar isochrones in Appendix 2, spanning from 250M years to 850M years.

Note: to facilitate your plotting, the isochrones provided in Appendix 2 have been truncated: stars below 1 solar mass and stars that have evolved past the main sequence are **NOT** included in the subsequent isochrones.

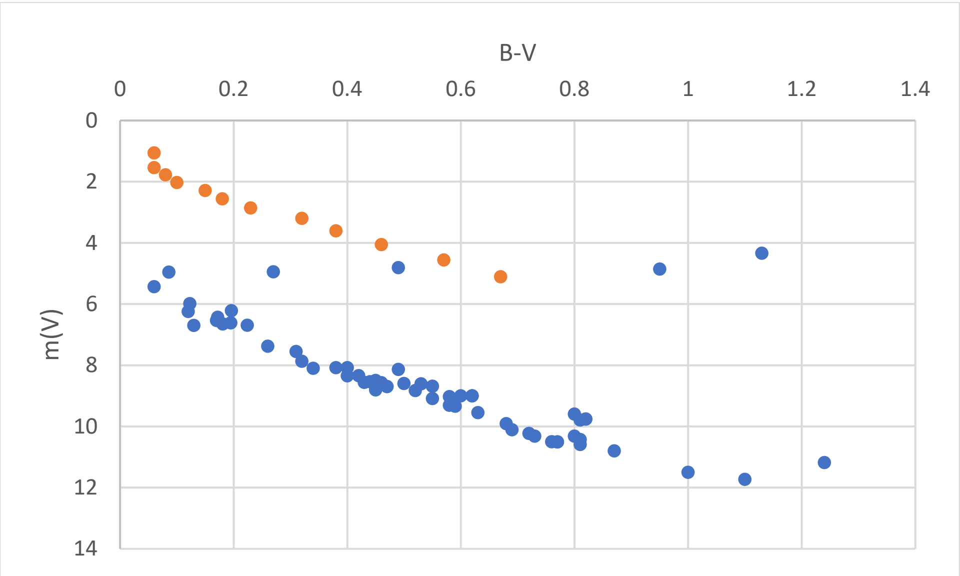
viii) Given the information above and the set of theoretical isochrones in Appendix 2, describe a simple way of determining the best fit isochrone. [1 mark]

Simply compare B-V values for the main-sequence turnoff, since these are supposedly corrected for interstellar extinction. Pick the isochrone that has the most similar B-V value for the turnoff point.

ix) Implement the procedure stated in part 7. On the separate diagram provided, clearly plot the best fit isochrone to the Coma Star Cluster CMD. [2 marks].

x) Hence or otherwise, estimate the age of the Coma Star Cluster. Also, re-estimate the distance to the Coma Star Cluster. Clearly state all assumptions used, if any. [2 marks]

Best fit isochrone: 550m years.



Acceptable range for distance modulus u= (4.39 ± 0.2)

$$d = 10^{\frac{4.39}{5}+1} \approx 76\ pc$$

xi) Suppose that we used the uncorrected CMDs instead. How would our estimates differ from the true value? Briefly explain (computations are not required). [2 marks]

The cluster will appear redder and dimmer. This causes the estimated distance to be larger than the true distance, while the estimated age of the cluster will be larger than the true cluster age.

Appendix I: ZAMS data

Note: Mass, L and R are expressed in terms of the solar units Msun, Lsun and Rsun respectively. Age stands for the age of the star (in years) at the point it hits the main sequence, while ST stands for Spectral Type. Mbol and Mv stand for the absolute bolometric and visual magnitudes respectively.

The ZAMS and isochrones were generated by Siess (1997)

Mass	L	R	Teff (K)	Age	Mbol	B-V	Mv	ST
0.5	4.63E-02	4.29E-01	3829	1.02E+08	8.09	1.43	9.32	M1
0.6	8.29E-02	5.22E-01	4008	8.32E+07	7.45	1.37	8.5	M0
0.7	1.48E-01	6.37E-01	4287	6.58E+07	6.82	1.21	7.57	K6
0.8	2.80E-01	7.56E-01	4698	5.18E+07	6.13	1	6.64	K4
0.9	4.88E-01	8.66E-01	5047	4.16E+07	5.53	0.86	5.91	K2
1	8.10E-01	9.97E-01	5334	3.31E+07	4.98	0.8	5.28	K0
1.1	1.30E+00	1.17E+00	5556	2.63E+07	4.47	0.73	4.71	G8
1.2	2.00E+00	1.33E+00	5788	2.19E+07	4	0.65	4.2	G5
1.3	2.95E+00	1.48E+00	6070	1.89E+07	3.58	0.56	3.75	G0
1.4	4.28E+00	1.66E+00	6284	1.60E+07	3.17	0.49	3.33	F7
1.5	6.07E+00	1.85E+00	6498	1.35E+07	2.79	0.41	2.93	F5
1.6	8.12E+00	1.97E+00	6770	1.17E+07	2.48	0.37	2.59	F3
1.7	1.09E+01	2.06E+00	7120	1.01E+07	2.16	0.32	2.26	F1
1.8	1.41E+01	2.13E+00	7476	8.87E+06	1.88	0.25	1.98	A9
1.9	1.78E+01	2.20E+00	7832	7.81E+06	1.62	0.2	1.74	A8
2	2.23E+01	2.27E+00	8176	6.94E+06	1.38	0.17	1.51	A7
2.2	3.33E+01	2.40E+00	8824	5.56E+06	0.94	0.08	1.13	A3
2.5	5.44E+01	2.51E+00	9705	4.14E+06	0.41	-0.02	0.8	A0
2.7	7.34E+01	2.62E+00	10241	3.44E+06	0.09	-0.04	0.57	A0
3	1.06E+02	2.72E+00	11048	2.68E+06	-0.31	-0.07	0.32	B9

3.5	1.84E+02	2.94E+00	12094	1.82E+06	-0.91	-0.09	-0.09	B8
4	3.17E+02	3.24E+00	13208	1.27E+06	-1.5	-0.11	-0.45	B7
5	6.82E+02	3.53E+00	15334	6.87E+05	-2.33	-0.14	-0.89	B6
6	1.34E+03	3.91E+00	17291	3.99E+05	-3.07	-0.17	-1.32	B4
7	2.37E+03	4.32E+00	19281	2.46E+05	-3.69	-0.2	-1.62	B3

Appendix II: Stellar Isochrones (250M years to 850M years)

Note: Mass, L and R are expressed in terms of the solar units Msun, Lsun and Rsun respectively. ST stands for Spectral Type. Mbol and Mv stand for the absolute bolometric and visual magnitudes respectively.

250M years

Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.58E-01	8.97E-01	5730	4.92	0.67	5.13	G6
1.1	1.36E+00	1.01E+00	6047	4.41	0.57	4.59	G0
1.2	2.08E+00	1.14E+00	6360	3.96	0.46	4.11	F7
1.3	3.05E+00	1.25E+00	6675	3.54	0.38	3.66	F4
1.4	4.30E+00	1.33E+00	7055	3.17	0.33	3.27	F1
1.5	5.90E+00	1.36E+00	7569	2.82	0.23	2.93	A9
1.6	7.84E+00	1.39E+00	8051	2.51	0.18	2.64	A7
1.7	1.02E+01	1.43E+00	8488	2.23	0.11	2.39	A4
1.8	1.29E+01	1.49E+00	8865	1.97	0.07	2.16	A3
1.9	1.62E+01	1.55E+00	9202	1.72	0.03	1.96	A2
2	2.01E+01	1.62E+00	9499	1.49	-0.01	1.83	A1
2.2	3.02E+01	1.78E+00	10012	1.05	-0.03	1.49	A0
2.5	5.31E+01	2.08E+00	10619	0.44	-0.05	1	B9
2.7	7.53E+01	2.36E+00	10890	0.06	-0.06	0.66	B9
3	1.25E+02	3.01E+00	10943	-0.49	-0.06	0.12	B9

325M years

Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.62E-01	8.99E-01	5732	4.91	0.67	5.13	G6
1.1	1.37E+00	1.02E+00	6049	4.41	0.57	4.59	G0
1.2	2.09E+00	1.14E+00	6360	3.95	0.46	4.1	F6
1.3	3.08E+00	1.26E+00	6678	3.53	0.38	3.65	F4
1.4	4.38E+00	1.34E+00	7062	3.15	0.33	3.25	F1
1.5	5.99E+00	1.37E+00	7570	2.81	0.23	2.91	A9
1.6	7.99E+00	1.41E+00	8045	2.49	0.18	2.62	A7
1.7	1.04E+01	1.46E+00	8453	2.21	0.12	2.37	A5
1.8	1.33E+01	1.52E+00	8798	1.94	0.08	2.13	A3
1.9	1.67E+01	1.60E+00	9109	1.69	0.04	1.92	A2
2	2.08E+01	1.68E+00	9376	1.45	0.01	1.75	A1
2.2	3.17E+01	1.88E+00	9852	1	-0.02	1.41	A0
2.5	5.70E+01	2.32E+00	10296	0.36	-0.04	0.86	A0
2.7	8.24E+01	2.80E+00	10249	-0.04	-0.04	0.45	A0

400M years

Mass	**L**	**R**	**Teff**	**Bol**	**B-V**	**Mv**	**ST**
1	8.66E-01	9.00E-01	5733	4.91	0.67	5.12	G6
1.1	1.38E+00	1.02E+00	6050	4.4	0.57	4.58	G0
1.2	2.11E+00	1.14E+00	6363	3.94	0.46	4.09	F6
1.3	3.12E+00	1.26E+00	6681	3.51	0.38	3.64	F4
1.4	4.46E+00	1.35E+00	7077	3.13	0.33	3.23	F1
1.5	6.09E+00	1.38E+00	7570	2.79	0.23	2.89	A9
1.6	8.14E+00	1.43E+00	8021	2.47	0.18	2.6	A7
1.7	1.06E+01	1.49E+00	8411	2.19	0.13	2.34	A5
1.8	1.36E+01	1.56E+00	8739	1.92	0.09	2.1	A3
1.9	1.72E+01	1.65E+00	9021	1.66	0.05	1.87	A2
2	2.15E+01	1.76E+00	9263	1.42	0.03	1.67	A1
2.2	3.32E+01	2.01E+00	9620	0.95	-0.01	1.31	A0
2.5	6.11E+01	2.69E+00	9718	0.29	-0.02	0.67	A0

475M years

Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.70E-01	9.02E-01	5734	4.9	0.67	5.12	G6
1.1	1.39E+00	1.02E+00	6051	4.39	0.57	4.57	G0
1.2	2.14E+00	1.15E+00	6366	3.93	0.46	4.08	F6
1.3	3.16E+00	1.27E+00	6685	3.5	0.38	3.62	F4
1.4	4.50E+00	1.35E+00	7082	3.12	0.32	3.22	F1
1.5	6.20E+00	1.40E+00	7572	2.77	0.23	2.88	A9
1.6	8.28E+00	1.45E+00	8001	2.45	0.18	2.58	A7
1.7	1.08E+01	1.52E+00	8375	2.16	0.13	2.31	A5
1.8	1.39E+01	1.61E+00	8662	1.89	0.1	2.07	A4
1.9	1.77E+01	1.72E+00	8914	1.63	0.07	1.83	A3
2	2.23E+01	1.84E+00	9114	1.38	0.04	1.6	A2
2.2	3.48E+01	2.18E+00	9369	0.9	0.01	1.19	A1
2.5	6.88E+01	3.14E+00	9258	0.16	0.03	0.41	A1

550M years

Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.74E-01	9.04E-01	5735	4.9	0.67	5.11	G6
1.1	1.40E+00	1.02E+00	6054	4.39	0.57	4.56	G0
1.2	2.16E+00	1.15E+00	6370	3.92	0.46	4.06	F6
1.3	3.20E+00	1.28E+00	6689	3.49	0.38	3.61	F4
1.4	4.57E+00	1.36E+00	7084	3.1	0.32	3.2	F1
1.5	6.31E+00	1.41E+00	7564	2.75	0.23	2.86	A9
1.6	8.43E+00	1.47E+00	7974	2.43	0.18	2.56	A7
1.7	1.10E+01	1.56E+00	8301	2.14	0.15	2.29	A6
1.8	1.43E+01	1.66E+00	8594	1.86	0.1	2.03	A4
1.9	1.82E+01	1.79E+00	8796	1.6	0.08	1.78	A3
2	2.31E+01	1.95E+00	8947	1.34	0.06	1.54	A3
2.2	3.63E+01	2.42E+00	9003	0.85	0.06	1.06	A2

625M years

Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.78E-01	9.05E-01	5736	4.89	0.67	5.11	G6
1.1	1.41E+00	1.03E+00	6056	4.38	0.57	4.55	G0
1.2	2.18E+00	1.16E+00	6372	3.91	0.45	4.05	F6
1.3	3.24E+00	1.28E+00	6693	3.47	0.38	3.6	F4
1.4	4.64E+00	1.37E+00	7085	3.08	0.32	3.18	F1
1.5	6.41E+00	1.42E+00	7551	2.73	0.24	2.84	A9
1.6	8.59E+00	1.50E+00	7952	2.42	0.18	2.54	A7
1.7	1.13E+01	1.59E+00	8252	2.12	0.16	2.26	A6
1.8	1.46E+01	1.72E+00	8507	1.84	0.11	2	A4
1.9	1.88E+01	1.87E+00	8653	1.57	0.1	1.74	A4
2	2.39E+01	2.07E+00	8747	1.3	0.09	1.48	A3
2.2	3.77E+01	2.75E+00	8507	0.81	0.11	0.97	A4

700M years

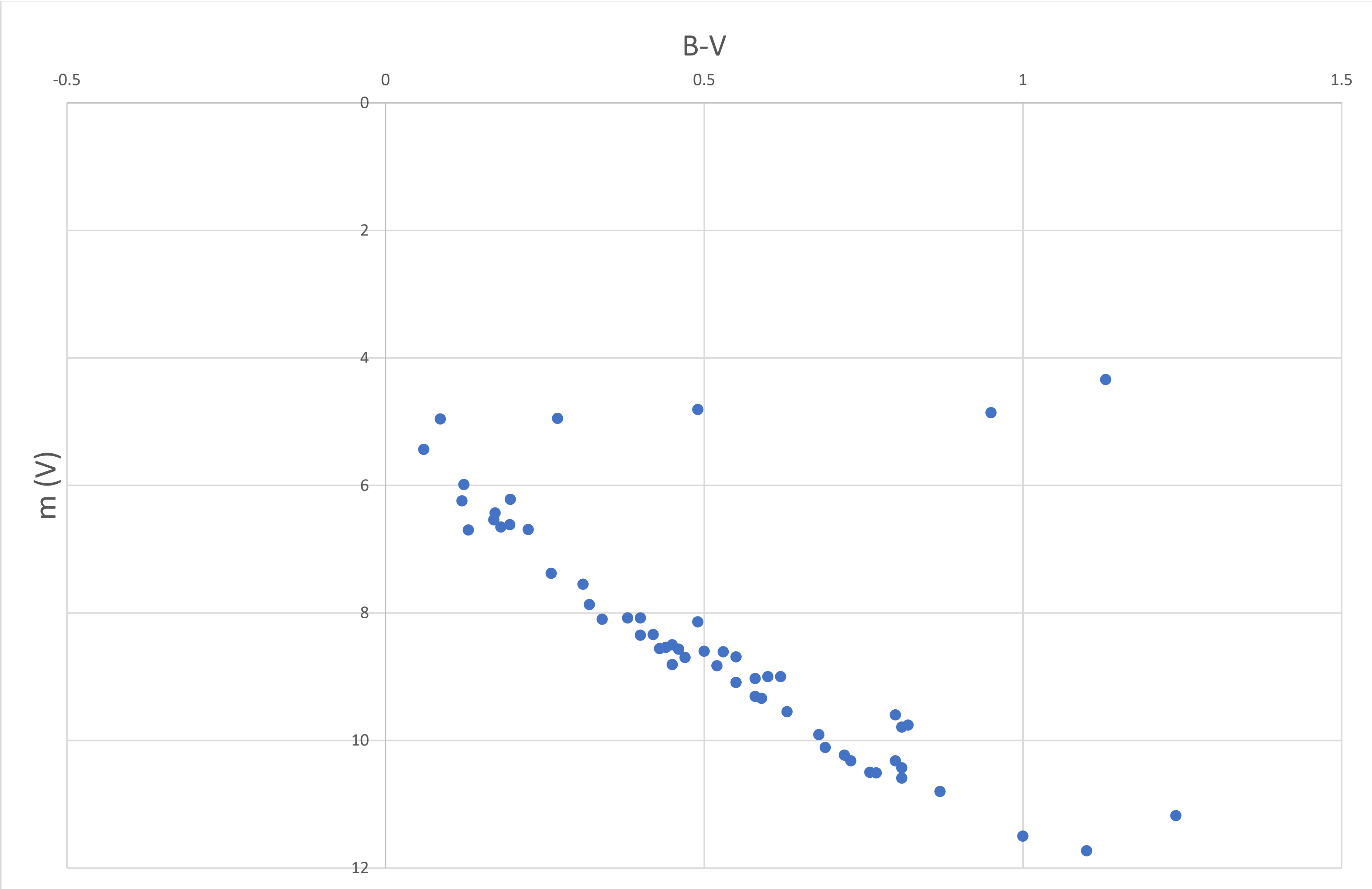
Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.83E-01	9.07E-01	5738	4.89	0.67	5.1	G6
1.1	1.42E+00	1.03E+00	6058	4.37	0.57	4.55	G0
1.2	2.20E+00	1.16E+00	6375	3.89	0.45	4.04	F6
1.3	3.28E+00	1.29E+00	6696	3.46	0.38	3.58	F4
1.4	4.71E+00	1.38E+00	7092	3.07	0.32	3.17	F1
1.5	6.50E+00	1.44E+00	7543	2.72	0.24	2.82	A9
1.6	8.73E+00	1.52E+00	7907	2.4	0.19	2.52	A7
1.7	1.15E+01	1.64E+00	8185	2.1	0.16	2.23	A7
1.8	1.50E+01	1.78E+00	8396	1.81	0.13	1.96	A5
1.9	1.93E+01	1.97E+00	8484	1.54	0.11	1.7	A4
2	2.47E+01	2.24E+00	8490	1.27	0.11	1.43	A4

775M years

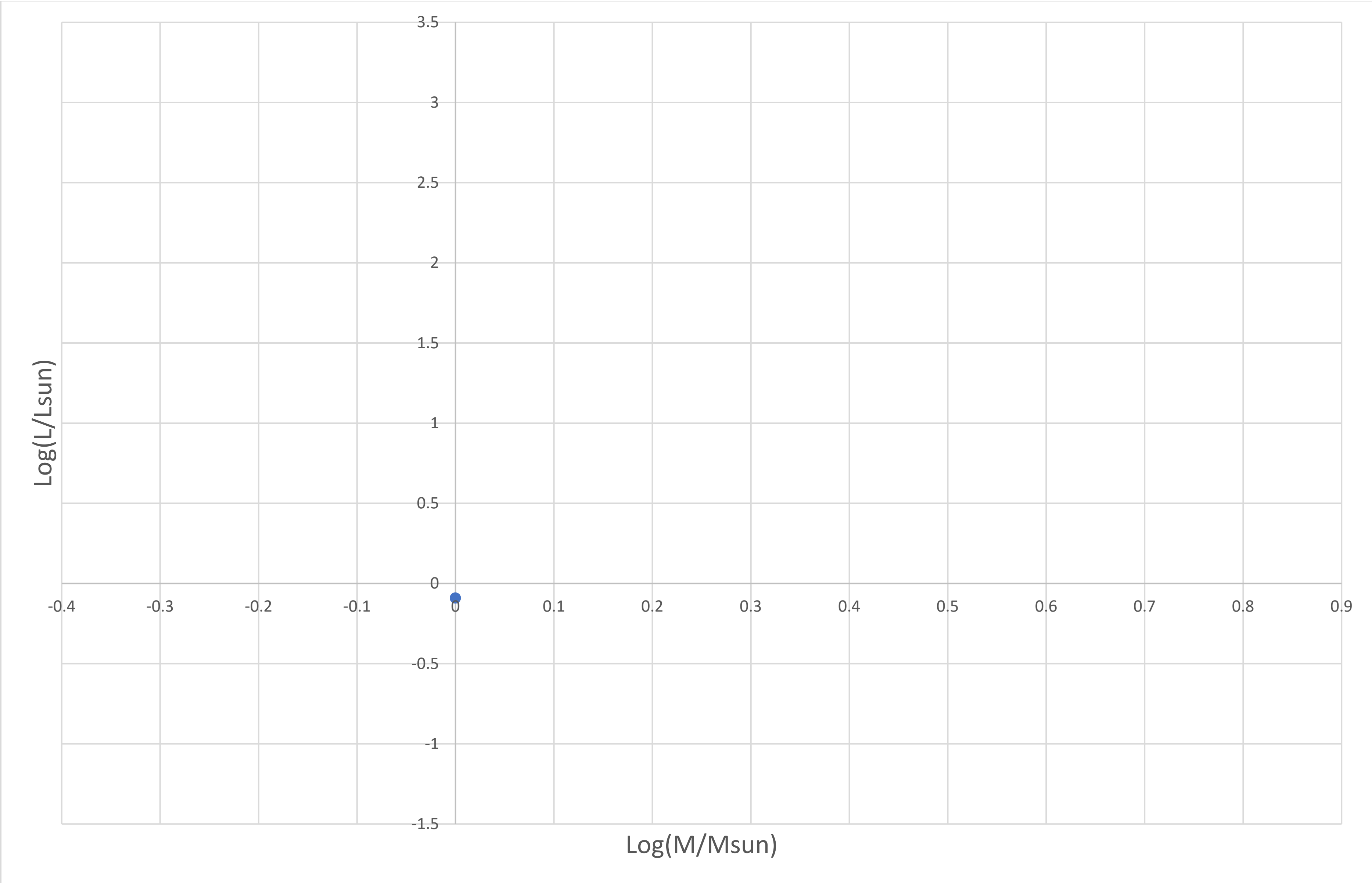
Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.87E-01	9.09E-01	5741	4.88	0.67	5.1	G6
1.1	1.43E+00	1.03E+00	6061	4.36	0.57	4.54	G0
1.2	2.22E+00	1.17E+00	6379	3.88	0.45	4.03	F6
1.3	3.32E+00	1.30E+00	6700	3.45	0.38	3.57	F4
1.4	4.78E+00	1.39E+00	7084	3.05	0.32	3.15	F1
1.5	6.61E+00	1.46E+00	7517	2.7	0.24	2.8	A9
1.6	8.91E+00	1.55E+00	7864	2.38	0.19	2.49	A7
1.7	1.18E+01	1.69E+00	8106	2.07	0.17	2.2	A7
1.8	1.54E+01	1.86E+00	8260	1.78	0.15	1.92	A6
1.9	1.99E+01	2.10E+00	8299	1.51	0.15	1.65	A6
2	2.53E+01	2.45E+00	8140	1.24	0.17	1.37	A7

850M years

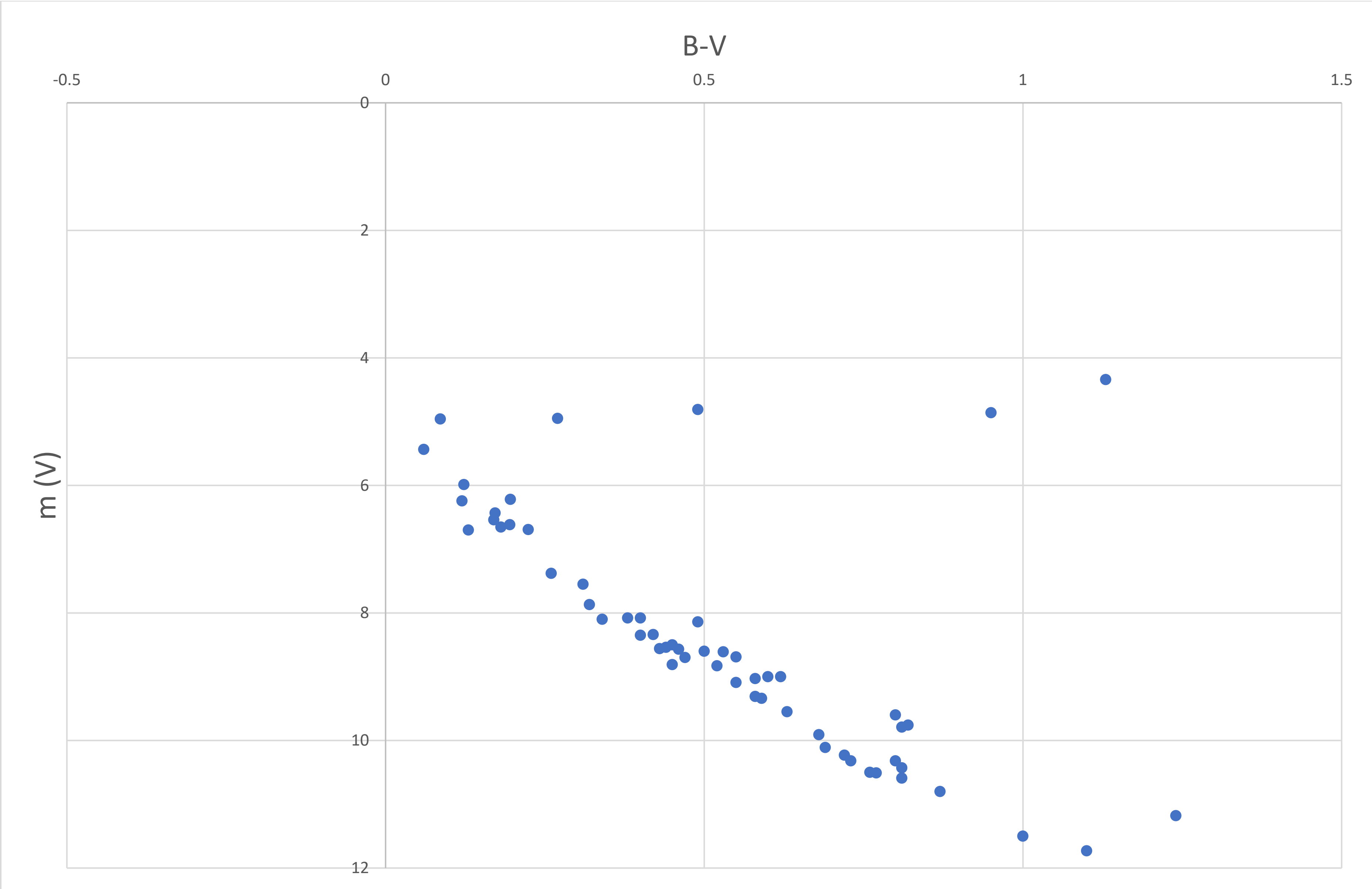
Mass	L	R	Teff	Bol	B-V	Mv	ST
1	8.92E-01	9.11E-01	5743	4.87	0.67	5.09	G6
1.1	1.44E+00	1.04E+00	6065	4.35	0.57	4.53	G0
1.2	2.25E+00	1.17E+00	6376	3.87	0.45	4.02	F6
1.3	3.37E+00	1.30E+00	6703	3.43	0.38	3.55	F4
1.4	4.85E+00	1.40E+00	7085	3.04	0.32	3.13	F1
1.5	6.71E+00	1.48E+00	7491	2.68	0.25	2.79	A9
1.6	9.06E+00	1.59E+00	7817	2.36	0.2	2.47	A8
1.7	1.20E+01	1.74E+00	8016	2.05	0.18	2.18	A7
1.8	1.58E+01	1.94E+00	8118	1.76	0.17	1.89	A7
1.9	2.04E+01	2.25E+00	8045	1.48	0.18	1.6	A7
2	2.67E+01	2.71E+00	7825	1.18	0.2	1.3	A8



B-V
-0.5
0
0.5
1
1.5
0
2
4
6
8
10
12
m (V)

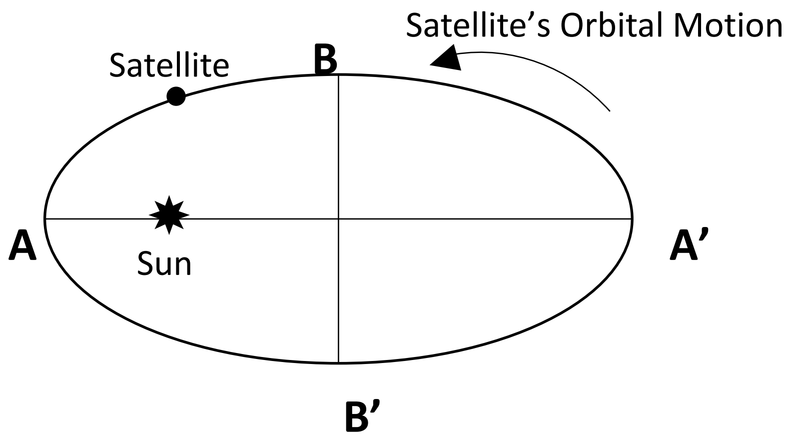


3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Log(L/Lsun)
Log(M/Msun)


B-V
-0.5
0
0.5
1
1.5
0
2
4
6
8
10
12
m (V)

DRQ 5: Elliptical Orbit Analysis [18 marks]

A satellite moves in an elliptical orbit with semi mayor axis $\boldsymbol{a}$, and eccentricity $\boldsymbol{e}$. The satellite can be considered as a particle that moves under the influence of Sun's gravity. Its velocities at perihelion and aphelion are denoted by V_{pe} and V_{ape}, respectively.



Based on the information above, please answer the following questions:

i) State 2 quantities that are conserved in the satellite's motion. [1 marks]

ii) Hence or otherwise, show that the velocity of the satellite at any arbitrary distance $\boldsymbol{d}$ from the sun along its orbits is as follows: [3 marks]

$$V_d{}^2 = 2GM_{sun} * \left(\frac{1}{d} - \frac{1}{2a}\right)$$

iii) Derive the expression for perihelion distance d_{pe} in the terms of $\boldsymbol{a}$, the perihelion velocity V_{pe}, and the aphelion velocity V_{ape}. [3 marks]

iv) State Kepler's 2nd Law [1 mark]

v) In order to simplify our work, we define here the area constant to be the constant of proportionality of the area to the period of orbit. It is known that the area constant is defined as half of the satellite's angular momentum per its unit mass at any arbitrary position, express the satellite's area constant in terms of the other known quantities. [2 marks]

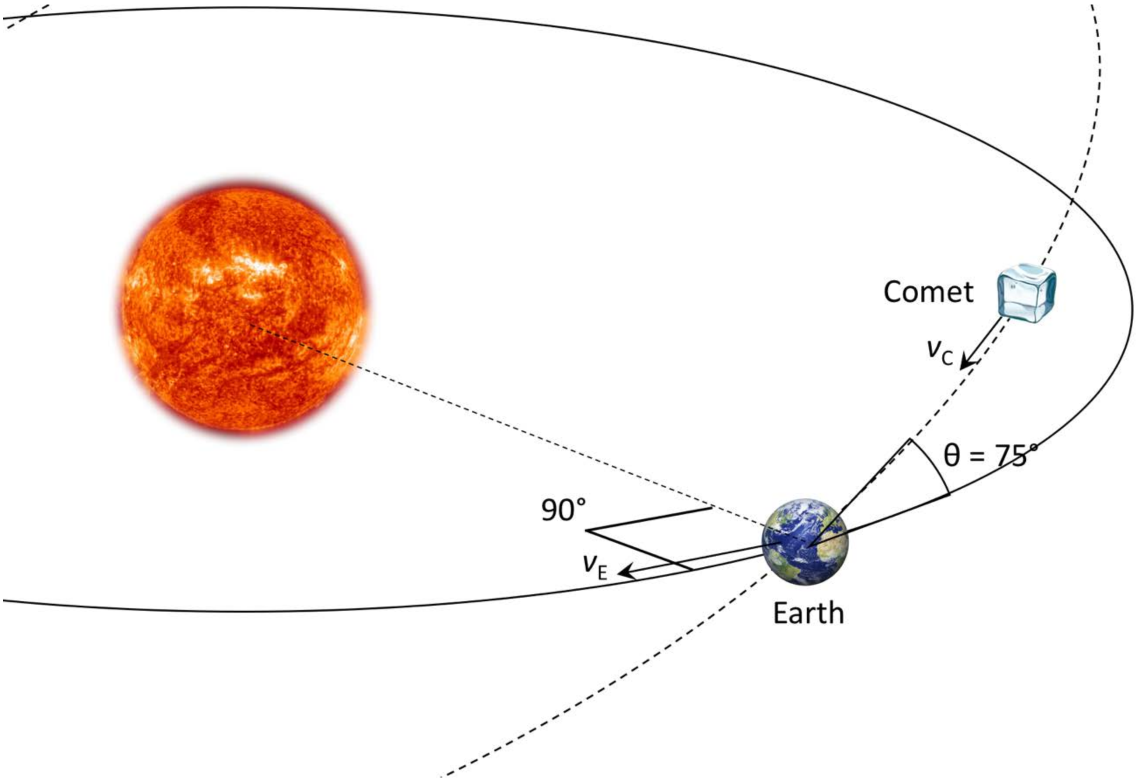
vi) Hence or otherwise, using your results in d) and e), prove that the orbital period P can be expressed as follows. [2 marks]

$$P = \frac{\pi a^2\sqrt{1-e^2} * (V_{pe} + V_{ape})}{V_{pe} * V_{ape}}$$

Recall that the area of an ellipse = $\pi ab = \pi a^2\sqrt{1-e^2}$, where b is the semi-minor axis.

vii) Using the results found above, calculate the velocities at perihelion and aphelion, $\boldsymbol{V_{pe}}$ and $\boldsymbol{V_{ape}}$ for Earth. Comment on the two values obtained. [2 marks]

We will now briefly study the legendary dinosaur killer, Chicxulub impactor. We will employ a model in which it is a massive metallic object with a mass of 5.3×10^{15}kg with a kinetic energy before impact of about 0.7×10^{24}J. In this model, it's believe the impactor was originally a long-period comet in orbit about the sun.



viii) Calculate the angular momentum of the impactor just before impact assuming that it collided tangentially to Earth along its orbit. You may assume circular orbit for Earth here. [2 marks]

ix) Hence or otherwise, calculate the inclination of the final orbit relative to the original orbit should a similar object were to collide Earth in this day. Assume that the impactor approaches at an inclination of about 75.0° before impact, as measured from the plane of orbit of Earth. [2 marks]

Answer:

i) Energy of the satellite is conserved. As well as its angular momentum.

ii)

K_d = Kinetic Energy at arbitrary distance d, from the Sun

U_d = Gravitational potential energy at arbitrary distance d, from the Sun

K_a = Kinetic Energy at distance equals to the length semi major axis a, from the Sun

U_a = Gravitational potential energy at distance equals to the length semi major axis a, from the Sun

m = satellite's mass

M = Sun's mass

Based on law of conservation of energy:

$$K_d + U_d = K_a + U_a$$

$$0.5mV_d{}^2 - \frac{GM_{Sun}m}{d} = 0.5mV_a{}^2 - \frac{GM_{Sun}m}{a}$$

$$0.5mV_d{}^2 - \frac{GM_{Sun}m}{d} = 0.5m * \left(\frac{GM_{sun}}{a}\right) - \frac{GM_{Sun}m}{a}$$

$$0.5mV_d{}^2 - \frac{GM_{Sun}m}{d} = -\frac{GM_{Sun}m}{2a}$$

$$V_d{}^2 = 2GM_{sun} * \left(\frac{1}{d} - \frac{1}{2a}\right)$$

iii) Based on law of conservation of angular momentum:

$$d_{pe} * (mV_{pe}) = d_{apo} * (mV_{ape})$$

$$d_{pe} * (mV_{pe}) = (2a - d_{pe}) * (mV_{ape})$$

$$d_{pe} * V_{pe} = \left[\frac{2a * V_{ape}}{V_{pe} + V_{ape}}\right] * V_{pe}$$

$$d_{pe} = \left[\frac{2a * V_{ape}}{V_{pe} + V_{ape}}\right]$$

iv) Based on Kepler's 2nd law of planetary motion, the line joining the orbiting body and the barycenter sweeps constant orbital plane area as the body revolves at its orbit during a fixed amount of time.

v)

L_d = satellite's angular momentum at any arbitrary distance d, from the Sun

Area constants = $0.5 * \frac{L}{m} = 0.5 * d * \frac{mv_d Sin\ \theta}{m} = 0.5 * d * v_d\ Sin\ \theta$

where $L = r \times p = d \times (m\, v_d) = m\,(d \times v_d) = m\, v_d\, d\, Sin\ \theta$

vi) Based on Kepler's 2nd law on planetary motion, it can be deduced that:

$$Total\ Area\ Swept = Total\ time\ duration * Area\ constants$$

If the total time duration is the orbital period P itself, the total area swept is equal to the area of satellite's elliptical orbital area:

$$Ellipse\ area = P * Area\ constants$$

$$\pi a^2\sqrt{1-e^2} = P * (0.5 * d * V_d\ Sin\ \theta)$$

By choosing conditioning satellite at perihelion distance d_{pe} and substituting the answer obtained in sub-question a), the following relation can be obtained for $\theta = 90°$ at the perihelion.:

$$\pi a^2\sqrt{1-e^2} = P * \left(0.5 * \frac{2a * V_{ape}}{V_{pe} + V_{ape}} * V_{pe}\right)$$

$$P = \frac{\pi a\sqrt{1-e^2} * \left(V_{pe} + V_{ape}\right)}{V_{pe} * V_{ape}}$$

vii) From a), V_{pe} and V_{ape} can be respectively obtained when $d = a(1-e)$ and $d = a(1+e)$.

$$V_{pe} = \sqrt[2]{\frac{GM(1+e)}{a(1-e)}}$$

$$V_{ape} = \sqrt[2]{\frac{GM(1-e)}{a(1+e)}}$$

$$V_{pe} = 30{,}365\ m/s$$

$$V_{ape} = 29{,}366.9\ m/s$$

Which is to be expected since the eccentricity of earth's orbit is very small.

viii) Linear Momentum of Impactor = $\sqrt{0.7 * 10^{24} * 2 * 5.3 * 10^{15}} = 8.61 * 10^{19}\ kg\ m/s$

Angular Momentum of Impactor = $8.61 * 10^{19} * 1.496 * 10^{11} = 1.29 * 10^{31} kgm^2/s$

ix) $\text{Angular Momentum of Earth} = 1.496 * 10^{11} * 30364.46 * 6 * 10^{24}$

$$= 2.726 \times 10^{40} kgm^2/s$$

$$\text{ArcTan}[\frac{(1.289\times10^{31}) * \text{Sin}[75\text{Degree}]}{(2.726\times10^{40}) + 1.289\times10^{31}\text{Cos}[75\text{Degree}]}/]$$

$$= 4.57\times10^{-10\circ}$$

Which is next to nothing.

You may use linear momentum to calculate since at the point of impact, the distance between Sun and Earth = Sun and Impactor.