AstroChallenge 2023
Senior Team Round

SOLUTIONS

Monday 29th May 2023

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper consists of **37** printed pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your name, school, and team.
6. It is your responsibility to ensure that your answer script has been submitted.
7. The marks for each question are given in brackets in the right margin, like such: [**2**].

Question 1 Wish upon a shooting star

Part I Wait, it's all just rocks?

Is a shooting star really a star flying right above our heads? Well thank god no. If a star were to be within our Earth's atmosphere you wouldn't be alive to be doing this question now.

Then what is a shooting star? After you have confidently eliminated the possibility of what you are seeing being just a plane or a satellite, you might be looking at a meteor.

You may have heard of the terms meteorite, meteor and meteoroid and wondered are they just the same thing? And you'd be right (sort of). They refer to the same thing just under different conditions. A meteoroid is defined to be a rock in space within the size of 30 micrometers and 1 meter in diameter. A meteoroid becomes visible as a meteor when it intersects with the Earth's atmosphere at night. The ram pressure due to the collision between the meteoroid and the atmosphere heats up and ionises the material in the meteoroid, which glows as a result. If the meteoroid has the right trajectory and does not completely burn up in the atmosphere, it reaches the Earth's surface and becomes a meteorite. The meteorite may also form a crater due to the impact with the Earth's surface.

(a) Considering that the Earth is much bigger and older than the Moon, why do we observe more craters on the Moon? **[1]**

Solution:

Any 2 of the following reasons (0.5 marks each):

Earth has an atmosphere, but the Moon does not. Therefore most meteoroids heading towards the Earth are burned up in the atmosphere before they reach the Earth's surface.

Earth has weather, but the Moon does not. Weathering and soil erosion on Earth smoothens out the craters on the Earth's surface, but craters on the Moon remain for millions of years.

Earth has active plate tectonics while the Moon does not. This allows cratered crust to be recycled into the mantle, smoothing out the craters on the Earth's surface.

There are generally 3 main types of meteorites: stony, iron and stony-iron meteorites.

(a) **(b)** **(c)**

Figure 1: The three main types of meteorites: (a) stony (b) iron (c) stony-iron

Stony meteorites account for most meteorites, about 94%. These meteorites contain silicate material. Iron meteorites account for about 5% of meteorites. These meteorites are composed of iron-nickel alloys. Stony-iron meteorites account for the last 1% and are a mixture of both silicate material and iron-nickel alloys.

(b) Most meteorites are believed to originate from the early solar system. Given their origins, explain how the three types of meteorites might be formed during the early solar system. **[3]**

Solution:

Planetesimals in the earl solar system undergo mass differentiation, where denser compounds such as metals tend to sink to the core. [1]
Stony meteorites originate from the crust, while Iron meteorites originate from the core of planetesimals. [1]
Stony-Iron meteorites originate from the transition region between core and crust of planetesimals. [1]

Marker's note: Many answers talked about how the elements iron, silicon etc are formed and released in the life cycle of heavy mass stars. However, the question asked how the meteorites *formed* in the *early solar system*, not where the elements originated from.

(c) Why are you more likely to observe a meteor after local midnight? **[2]**

Solution:

After midnight, the observer is on side of earth facing forward direction of earth's orbit [1]

Either:
- Meteors are thus brighter due to higher entry velocity into the atmosphere on the forward side. [1]
or:
- Classical aberration; due to the earth's motion through space, sporadic meteors impact the forward side of the earth more frequently. [1]

Marker's note: Many answers explained how at midnight, the sky at zenith is the 'darkest' due to being at maximum angular separation from the Sun's position. However, the question asked for *after* local midnight, not at local midnight. Furthermore, the difference in contrast at midnight is not a significant factor affecting the visibility of meteors.

Part II Rock + Ice = Comet

If your shooting star remains at the same spot for a while then maybe you are looking at a comet instead.

A comet is a solar system body made out of rock and ice. When a comet moves close to the Sun in the comet's orbit, volatile materials such as ice in the comet vaporises due to solar radiation and wind. This is known as outgassing. Out-gassing also releases solid material (aka dust). The vaporised volatile material gets ionised by ultraviolet radiation from the Sun and emits faint blue light to form a visible gas (or ion) tail. The gas tail is highly influenced by the direction of solar wind. The dust forms a dust tail that becomes visible as the dust scatters light from the Sun. The dust then settles into its own orbit around the Sun. However, not all comets have 2 tails. A comet may have a dust tail, a gas tail, both tails or no tail depending on its composition and distance from the Sun.

[Turn over]

Figure 2: An artist's impression of a comet

(d) With reference to 2, state and explain which is the gas tail and which is the dust tail. **[3]**

Solution:

A is the dust tail and B is the gas tail. [1]

- **A: The dust tail is curved along the orbital path of the comet as it is largely follows the original orbit of the comet, and is only slightly perturbed by the solar wind/radiation pressure. [1]**
- **B: The gas tail points away from the Sun in the direction of the solar wind. Due to the low density of gas, force from the solar wind is dominant and the gas tail does not curve along the orbital path. [1]**

  Marker's note: While the answer could be inferred from the descriptive paragraph before the question, a direct copy of the paragraph is not awarded any marks even if marking points are present.

With all the outgassing and releasing of dust, a comet naturally has a finite lifespan. Assuming the comet does not get ejected out of the solar system or experience breaking up or collision with other solar system objects, it will eventually become an extinct comet once all the volatile material evaporates. The comet will then become just a rock that resembles an asteroid.

But since we still see comets today, there must be a source for all these comets.

Comets can generally be classified by their orbital period. One group of comets is known as long-period comets. These comets have periods of more than 200 years and have been observed at almost every inclination to the Earth's orbital plane. The designation of "C/" is used to indicate long-period comets.

The other group of comets is (of course) known as short-period comets. These comets have periods of less than 200 years. Notably, a subset of this group, the Jupiter family of comets, have periods of less than 20 years. These short-period comets have been observed to have little orbital inclination relative to the Earth's orbital plane. The designation of "P/" is used to indicate short-period comets.

Extra: "D/" is used to refer to dead or destroyed comets :(

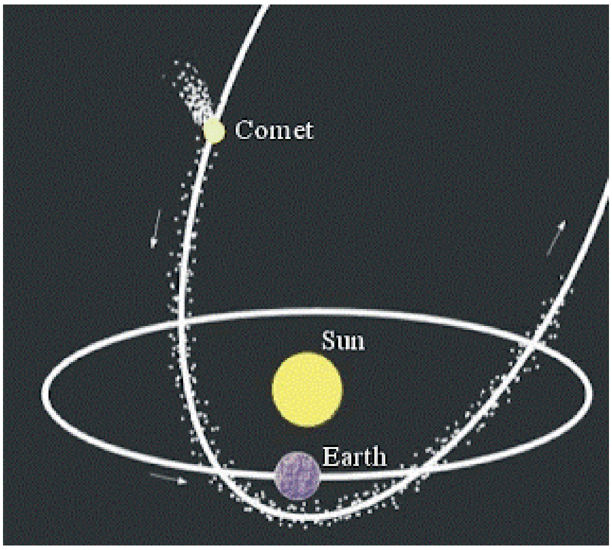


Figure 3: A diagram of a long-period comet orbit

(e) **Suggest a possible source for long-period comets and another source for short-period comets. Explain why.** [3]

Solution:

Long-period comets originate from the Oort cloud. [0.5]

- **The Oort Cloud has a large radii (2000-5000 AU), which explains why long-period comets have very large eccentricities and semi-major axes (and thus a long period). [0.5]**
- **The Oort Cloud is spherical with objects orbiting in random directions, which explains why long-period comets have a large variation in orbital inclinations. [0.5]**

Short-period comets originate from the Kuiper Belt/Centaurs/Scattered Disk. [0.5]

- **The Kuiper Belt/Centaurs/Scattered Disk are much closer to the Sun, which explains the short period of these comets. [0.5]**
- **The Kuiper Belt/Centaurs/Scattered Disk are concentrated along the ecliptic plane, which explains why short-period comets have less variation in orbital inclinations. [0.5]**

If you want to check how visible a comet is, you can do so with the aid of a diagram known as the brightness curve of a comet (4), which plots the apparent magnitude of the comet over time as the comet moves ahead in its trajectory around the sun. These plots can either real-time reports based on actual astronomical observations or forecasts based on theoretical comet models. We usually expect the peak brightness of a comet to happen near its closest approach to the sun at perihelion.

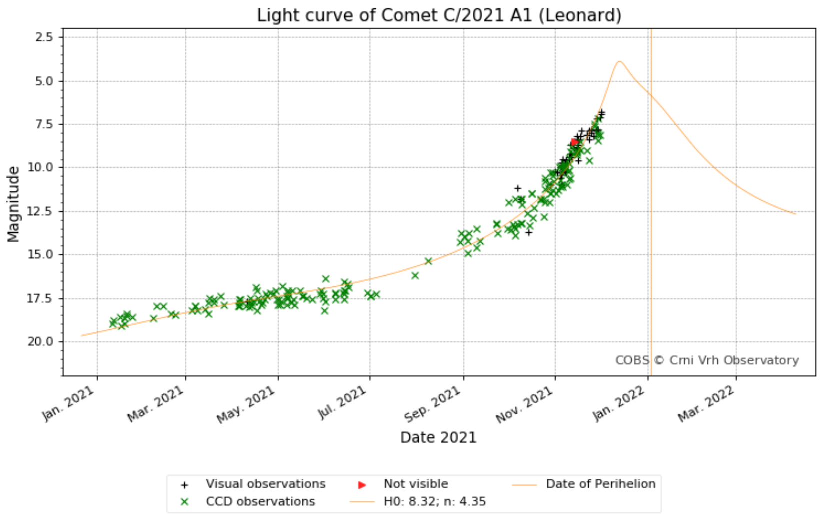


Figure 4: Brightness curve of comet C/2021 A1 Leonard

(f) Is it always the best time to view a comet when it is at peak brightness? Why or why not? **[2]**

Solution:

No, it depends on the relative position of the comet in the sky. If the comet is too near to the Sun as viewed from Earth (small angular separation with the Sun) [1], glare/illumination from the Sun can make it difficult to view the comet. [1]

Comment: Answers explicitly referencing the graph in Figure 4 only were not accepted as the question asks for a general answer applicable to all comets.

Part III It's raining stars

If you see stars raining from the sky, don't panic. Chances are it's not the end of the world and you might be looking at a meteor shower.

A meteor shower is an event where many meteors appear to originate from 1 point in the sky. Remember how a comet releases dust during outgassing as it travels near the sun? The dust released mostly stays on the orbital path of the comet. If the Earth's orbital path intersects with the comet's orbital path, once a year the Earth will be 'spinning through' this cloud of dust. This causes many meteoroids to be present in the Earth's atmosphere at the same time, resulting in a meteor shower.

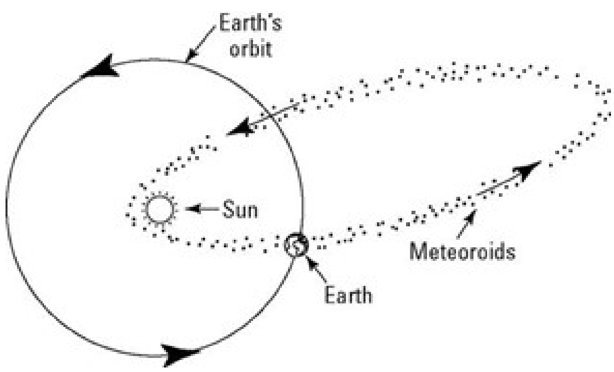


Figure 5: Trajectory of meteoroids in the comet's wake

All meteor showers originate from a radiant point. This means that if we extend the lines of the visible meteors, the lines will sort of connect at a small region as shown below. Meteor showers are also named based on their radiant point. For example, the Perseids meteor shower is named as such because its radiant point is in the constellation Perseus.

Figure 6: Meteors appear to origin radially from a single point in the sky

(g) Why do meteor showers originate from a radiant point? (Hint: Think about train tracks) **[2]**

Solution:

Members of a meteor shower originate from the same meteor stream, and are thus travelling parallel in space. [1] The radiant is the vanishing point of the parallel trajectories of the meteors as seen from earth. [1]

Halley's comet has an orbital inclination of 162 degrees to Earth's orbit (meaning it is inclined 18 degrees relative to the plane of the Earth's orbit, and is orbiting in the opposite direction). Interestingly, both the Eta Aquarids and Orionids meteor showers originate from Halley's comet, and they occur roughly 6 months apart from each other.

(h) Given the orbital inclination of Halley's comet, explain why the Eta Aquarids and Orionids occur roughly six months apart. **[1]**

Solution:

The Eta Aquariids and Orionids occur during the time of the year when the Earth crosses the meteor stream left by Halley's comet along its orbit. [0.5]

The orbit of the comet lies in a plane centered about the sun. Since the orbital plane of the comet is inclined, it necessarily intersects the orbital plane of the earth along a straight axis passing through the sun.

Taking the Earth's orbit to be roughly circular, the orbital plane of the comet will necessarily intersect Earth's orbit at two opposite points, resulting in a 6 month separation from when the Earth reaches 1 point to the other point. [0.5]

Marker's note: Drawings/diagrams clearly indicating the 2 points of intersection between the comet's orbit and Earth's orbit (1) on opposite ends of the earth's orbit (2) with the Sun in the center (3) was awarded the point.

There are many meteor showers, but why have we mostly heard of a few of them such as the Geminids or the Perseids? This might have to do with how many meteors can be seen during the meteor shower. More meteors would make for a more spectacular meteor shower.

A way to quantify this 'niceness' of meteor shower is known as the Zenithal Hourly Rate (ZHR). Since the peak of a meteor shower is generally taken to be when the radiant point is near the meridian, ZHR refers to the number of meteors 'crossing' the zenith in one hour of peak activity. Hence comets that left behind a denser cloud of dust would result in a meteor shower with a greater ZHR.

(i) **Given that the Geminids (RA: 7h 28min) peaks around mid-December every year, estimate what is the time of its highest activity during peak season to the nearest hour. Give your answer in local solar time and briefly justify your answer.** [2]

Solution:

At local midnight, the RA of the local meridian on the following dates are:

- **Autumn equinox (23rd September): 0000hrs**
- **Winter solstice (21st December): 0600hrs**

Mid-December is roughly a week earlier than Winter Solstice, which corresponds to an RA 30 mins earlier than the RA at Winter Solstice. Thus estimated RA is 5hrs 30mins. [1]

The RA of the radiant point of the Geminids is 7hrs 28mins, which is roughly 2 hours after local midnight in mid-December. Hence, time of peak activity is approximately 2am local solar time. [1]

(j) **The Leonids experiences a large spike in ZHR every 33 years, causing a meteor storm. Suggest why this happens.** [1]

Solution:

Every 33 years, the source comet reaches the point in its orbit that intersects the Earth's orbit around the Sun. [0.5]

The meteor stream is thus renewed with more new material left by the comet every 33 years. [0.5]

Well now you know. The next time you see a shooting star, remember that they are all space rocks and ice in the midst of their celestial journey and you just happened to be facing the right direction in the sky at the right time to catch a glimpse of it. And remember to make a wish, it probably works just as well whether its stars or space rocks.

Question 2 Twinkle Twinkle Little Star in the Great Nebula

Part I Twinkle Twinkle...

Since antiquity, humanity has been looking up into the night sky. Mesmerised by the shiny objects above our heads, we have studied and classified them extensively.

(a) It can be observed that most stars twinkle in the sky, except for a few "stars" that wander across the sky rapidly. Explain why some "stars" twinkle, and why wandering "stars" do not. [1]

Solution:

The wandering "stars" are planets. They are near Earth and cannot be treated as point sources of light. The other stars are much further, so they can be treated as point sources of light. The random scattering of light in the atmosphere causes significant flickering of the light of the very distant stars, but is insufficient to cause all light rays from the disk of the planets to be scattered beyond observation. [1]

Some other "stars" "twinkle" with a much longer period than the case in (a). Examples of which include photometric binaries and variable stars.

(b) Explain why the apparent magnitude of a photometric binary varies periodically. [1]

Solution:

A photometric binary is a binary system of stars with close proximity such that they cannot be resolved by observers on Earth.

However, the pair of stars may eclipse each other, if the observer is looking at the star along the plane of the orbit of the binary, causing the variation of apparent magnitude. [1]

Part II ... Little Star ...

Before delving deeper into variables, we need to have a thorough understanding of stars. In modern astronomy, we believe that stars must also obey the laws of physics. We can model stars with a few physical principles. One of the principles is that of mass continuity, described by the equation below.

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \tag{1}$$

where r is the radial distance from the centre of a star. $M(r)$ describes the mass of the portion of the star contained within a sphere with a radius of r centred at the centre of the star, and $\rho(r)$ is the density of the star at a radial distance r away from the centre of the star. The equation can be understood as for a small increase in a distance dr, the increase in mass dM is equal to a thin spherical shell of radius r and density $\rho(r)$.

A physically sensible model of a star will dictate that $M(r = 0) = 0$ (the center of the star starts of with 0 mass) and $M(r \geq R) = M_\star$ (there is no additional mass beyond the surface of the star). Here, R is the radius of the star and $M_\star$ is the mass of the star. These dictations are known as boundary conditions.

(c) What will the boundary conditions on $M(r)$ imply for the values of $\rho(r)$ for different values of r? [1]

Solution:

The density is positive for $r < R$ and is zero for $r \geq R$.

Another principle that is important in describing a star is that of hydrostatic equilibrium. Due to the self-gravity of stars, the pressure of the plasma within the star is highest near the centre and decreases radially outward. A star is in hydrostatic equilibrium if at all points in the star, the gravitational force experienced by the stellar matter is perfectly balanced by the pressure gradient.

The equation for hydrostatic equilibrium can be derived by considering a small rectangular disk of mass (Figure 7), with a base area of A and a height of dr, at a distance r from the centre of the star. The downward force (W) due to gravity and the upwards force (F) due to the pressure difference at the points below and above the disk is balanced at equilibrium.

(d) Derive the equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = \frac{GM(r)\rho(r)}{r^2}$$

Justify your steps in detail and quote all theorems you have used. **[3]**

Solution:

The mass of the disk is $\rho(r) \cdot A \cdot dr$ and the magnitude of gravitational field strength at r is $\frac{GM(r)}{r^2}$ given by Newton's shell method. So, the weight of the disk is $\frac{GM(r)}{r^2} \cdot \rho(r) \cdot A \cdot dr$. [1]

The pressure difference is $dP(r)$ and hence the upward force is $dP(r) \cdot A$. [1]

Balancing the forces (i.e. equating the two forces), we obtain $\frac{dP(r)}{dr} = \frac{GM(r)\rho(r)}{r^2}$. [1]

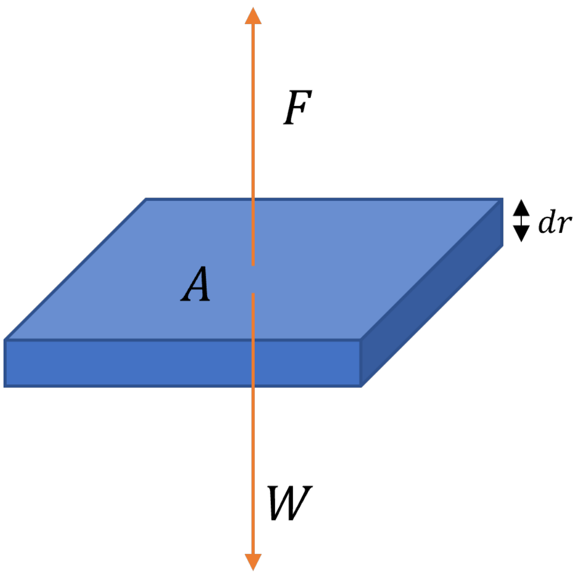


Figure 7: Disk under hydrostatic equilibrium.

Yet another principle is the all-familiar conservation of energy. It is formulated as below.

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r), \tag{2}$$

where $L(r)$ is the contribution to luminosity from an inner spherical portion of the star with radius r from the centre and $\epsilon(r)$ is the rate of energy production per unit mass of the star at the distance r from the centre.

(e) You are given the following integral:

$$\int_0^R 4\pi r^2 \rho(r)\epsilon(r) dr = 4\pi k R^2 \left[T_{surface}\right]^4,$$

where $T_{surface}$ is temperature of the star at its surface (r = R). Explain the meaning of the integral on the left-hand side and state the value of k. [2]

Solution:

The integral on the left is proportional to the total luminosity of the star. [1]

Observe the indices of the factors on the right. This is essentially Stefan-Boltzmann Law. $k = 5.67 \times 10^{-8}$ **is the Stefan-Boltzmann constant.** [1]

Now, we are ready to construct a simple model of a variable star. The oscillations of a pulsating variable can be modeled by a standing sound wave established in the star. This is akin to the standing sound wave established in a pipe closed at one end; the closed end is analogous to the centre of the star and the open end the surface of the star. We can estimate the period of oscillation of the star, τ, by equating it to ***twice*** the time for the sound wave to travel the ***diameter*** of the star. The speed of sound in the star is given by the equation below if the sound wave is a result of adiabatic expansion and compression.

$$v = \sqrt{\frac{\gamma P}{\rho}}, \tag{3}$$

where γ is a constant. For simplicity and with a blatant disregard for reality, **assume** $\rho(r) = \rho$ **is constant**.

(f) Solve the equation of hydrostatic equilibrium to obtain $P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$.

Hint: what is a sensible value for $P(R)$ (pressure at the surface r=R)? [3]

Solution:

The constant density assumption gives $M(r) = \frac{4}{3}\pi r^3\rho$. This reduces the hydrostatic equilibrium equation to $\frac{dP(r)}{dr} = -\frac{4}{3}\pi G\rho^2 r$. [1]

There should not be pressure at the surface of the star, so $P(R) = 0$. [1]

$$\int_r^R \frac{dP(r)}{dr}dr = -\frac{4}{3}\pi G\rho^2 \int_r^R r dr.$$

Solving this directly gives $P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$. [1]

We will now attempt to derive the equation for the period of oscillation of the star. Since the time taken for a sound wave to travel a distance of dr is $dt = \dfrac{dr}{v(r)}$, we can construct the integral for the total time τ accordingly.

(g) Write down the integral for the total time τ for a sound wave to travel the diameter of a star, solve it and show that the period of oscillation is given by:

$$\tau = \sqrt{\frac{6\pi}{\gamma G\rho}}.$$

This will come in handy:

$$\int \frac{1}{\sqrt{R^2 - r^2}}dr = \sin^{-1}\left(\frac{r}{R}\right) + c.$$

[3]

Solution:

The time taken for the sound wave to travel a distance of dr is $\frac{dr}{v(r)}$.

The period τ is the time taken for a sound wave to travel back and forth once across the diameter of the star, therefore $\tau = 4\int_0^R \frac{1}{v(r)}dr$. [1]

Substituting $P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$ into the expression for $v(r)$, and then substituting $v(r)$ into the above integral, we obtain [1]**:**

$$\tau = 4\int_0^R \sqrt{\frac{3}{2\pi\gamma G\rho}} \frac{1}{\sqrt{R^2 - r^2}}dr$$

Using the identity given [1]**:**

$$\tau = 4 \cdot \sqrt{\frac{3}{2\pi\gamma G\rho}} \cdot \frac{\pi}{2} = \sqrt{\frac{6\pi}{\gamma G\rho}}$$

(h) If a typical classical Cepheid is five times more massive and has a diameter 50 times longer than our Sun, what is the period of its pulsation? Does the period agree with that of a typical classical Cepheid variable (1 to 50 days)? Take $\gamma = \frac{5}{3}$. [2]

Solution:

The Mass and Radii of our Sun can be obtained from the formula booklet to calculate the density ρ. Substituting the values for γ, ρ and G, we obtain $\tau \approx 20$ days. It is within the range of typical classical Cepheid periods of 1 to 50 days.

Unlike the equation for the period of oscillation, the relationship of the pulsating period with luminosity is difficult to study analytically. Nevertheless, the variations of the luminosity, the effective temperature, and the radius of δ Cephei (the archetypal classical Cepheid) are given in the figure below.

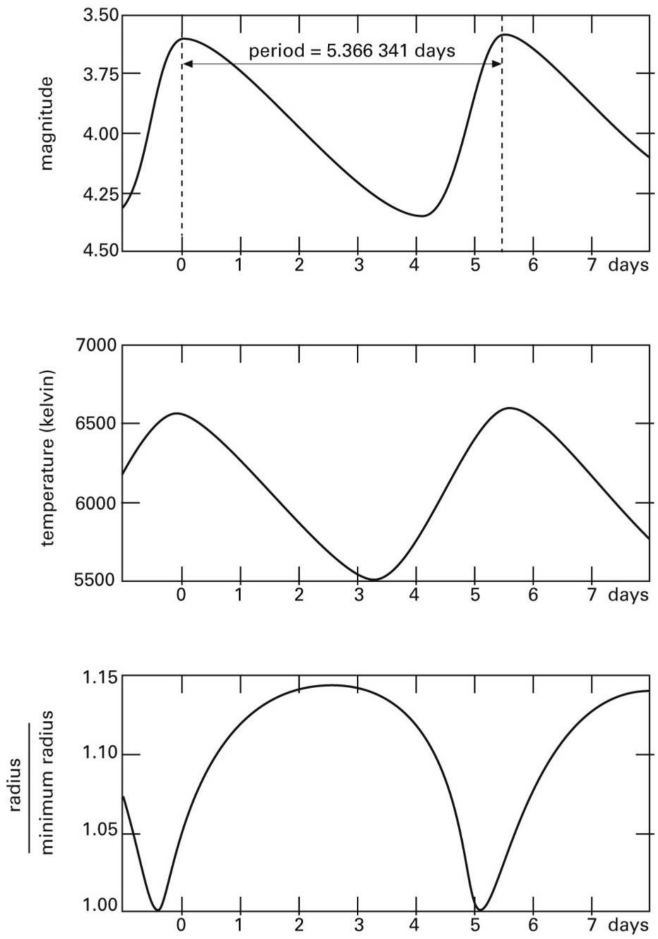


Figure 8: Properties of δ Cephei. Modified from The Free Dictionary by Farlex.

(i) With reference to Figure (8), state if temperature or radius has a greater impact on the variation of luminosity of δ Cephei. Briefly explain why is this the case.

Hint: study Equations (1) to (3). **[2]**

Solution:

Temperature has a greater impact on the luminosity variation of δ Cephei. [1]

By the Stefan-Boltzmann Law, luminosity is proportional to the fourth power of the temperature, but only to the square of radius, therefore it is more sensitive to changes in temperature rather than radius. [1]

Part III ... In the Great Nebula

First documented in the 10^{th} century by a Persian astronomer, the "Great Nebula" of Andromeda (Figure 9) is visible to the naked eye. It was studied as a nebula for many centuries, raising little suspicion. However, extraordinary characteristics of the "nebula" were gradually unveiled. It had a continuous spectrum with spectral lines similar to that of typical stars. It moved with a proper motion much more rapidly than many other objects in the night sky. The "novae" observed in the "nebula" was exceptionally faint. The nature of the "Great Nebula" became hotly debated near the 20^{th} century and was settled once and for all in the year 1925. By observing classical Cepheids in the "nebula", Edwin Hubble computed the distance of the "nebula" using a formula calibrated by Henrietta Swan Leavitt with the Cepheids in the Magellanic Clouds.

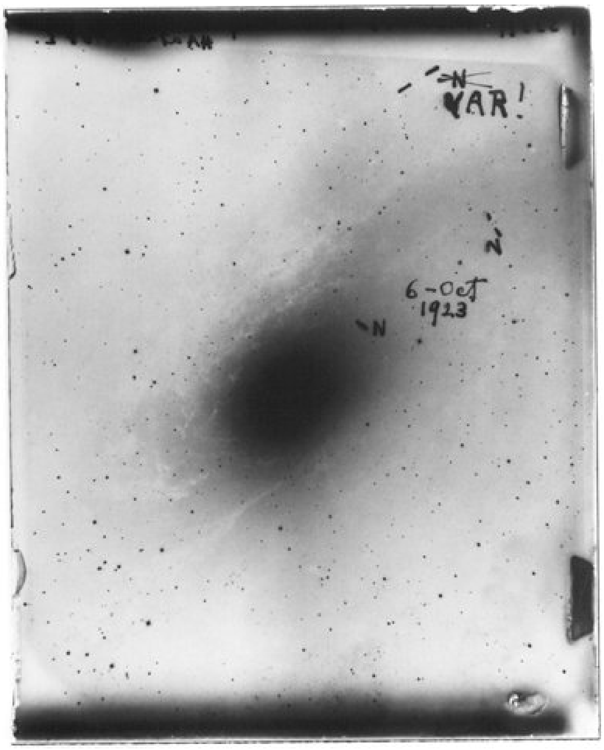


Figure 9: Edwin Hubble's photographic plate of Andromeda.

(j) Suppose a classical Cepheid in the "Great Nebula" of Andromeda has an average apparent magnitude of 18.7 and a period of 37 days. Estimate the distance to the "Great Nebula" of Andromeda. [2]

Solution:

By the luminosity-period relation, the absolute magnitude of the Cepheid is $M = -2.76\log(37) - 1.4 = -5.73$. **[1]**

The distance can be obtained with the distance modulus, $d = 10pc \times 10^{(18.7+5.73)/5} = 770kpc$. **[1]**

Question 3 Escape from the Solar System

Part I Earth-Jupiter Motion

When we observe Jupiter with the naked eye in the night sky, we notice that its motion relative to Earth is periodic. We can determine the period of this motion by assuming that the mass of the Sun M_s dominates and thus Jupiter and Earth only experience a gravitational force from the Sun, and that both Jupiter and Earth orbit in circular co-planar orbits.

(a) Given the above simplifying assumptions, calculate the orbital velocities of both Jupiter and Earth. Express your answers in km s^{-1}. **[1]**

Solution:

Assuming circular orbits, the gravitational force is equal to the centripetal force at an orbital distance r. [0.5]

$$F_c = \frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\mathbf{v} = \sqrt{\frac{GM}{r}}$$

Orbital velocity of Jupiter:

$$V_J = \sqrt{\frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{7.785 \times 10^{11}}} = 13.1 km/s \text{ [}\mathbf{0.25}\text{]}$$

Orbital velocity of Earth:

$$V_J = \sqrt{\frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{1.496 \times 10^{11}}} = 29.8 km/s \text{ [}\mathbf{0.25}\text{]}$$

Answers that calculate the orbital velocity from the orbital period were also accepted.

Due to the relative orbital velocities of Jupiter and Earth, Jupiter will return to the same position relative to the Earth and the Sun with a period S, which is also known as the synodic period. The synodic period of Jupiter is related to the orbital period of Jupiter (P_J) and Earth (P_E) by the following relation:

$$\frac{1}{S} = \frac{1}{P_E} - \frac{1}{P_J} \tag{4}$$

On 10 June 2019, Saitama, a nondescript bald man living in Japan, sees Jupiter reaching its highest altitude at midnight, which means that Jupiter is at opposition.

(b) How long will it take before Saitama is able to see Jupiter at opposition again? **[2]**

Solution:

The orbital period can be calculated from velocity by $v = \dfrac{2\pi r}{P}$.
Alternatively, orbital periods of Earth and Jupiter can be obtained from the formula booklet. Substituting into the equation for synodic period:

$$\frac{1}{S} = \frac{V_E}{2\pi r_E} - \frac{V_J}{2\pi r_J} \text{[1]}$$

$$\frac{1}{S} = \frac{29.8 \times 10^3}{2\pi \times 1.496 \times 10^11} - \frac{13.1 \times 10^3}{2\pi \times 7.785 \times 10^11}$$

$$\mathbf{S = 398.7 \text{ days}} \quad \text{[1]}$$

During the next Jupiter opposition, Saitama finds himself frustrated by the unbearably hot summer in Japan. To save the people of Japan from summer, he decides to send Earth on a trajectory away from the Sun with his overwhelming physical strength. Because he is lazy and wants to save as much energy as possible, Saitama decides to boost the speed of Earth by sending it on a gravitational slingshot trajectory around Jupiter.

A gravitational slingshot is generally used by spacecraft to reduce the fuel needed to accelerate or decelerate, and this is done by leveraging on the gravitational pull of a planet or some moving celestial body. The spacecraft performs a close flyby to the celestial body and uses its gravity to alter the speed and/or direction of its trajectory. This technique was used to boost the speed of the Voyager 1 and 2 spacecraft such that they could reach the velocity needed to reach an escape trajectory out of the solar system. We will now proceed to investigate the exact mechanism behind the gravitational slingshot.

Part II Gravitational Slingshot at Jupiter

At the moment of Jupiter's next opposition, Saitama carries out his gravitational slingshot plan by delivering an impulse to the Earth in a manner such that the Earth loses all of its tangential velocity, and instead gains a radial velocity v_0 equal to its initial orbital velocity around the Sun (Figure 10).

Since the mass of Jupiter is far greater than the mass of the Earth ($M_J >> M_E$), we can assume that as the Earth approaches Jupiter, its trajectory and motion relative to Jupiter is deflected by Jupiter's gravity in a manner similar to figure 13. To greatly simplify things, we shall also make the dubious (and probably catastrophic) assumption that the gravitational influence of the Sun can be ignored. By considering only the Earth-Jupiter interaction, we find that because Jupiter is also moving with an orbital velocity V_J, the Earth gains a velocity boost as a result.

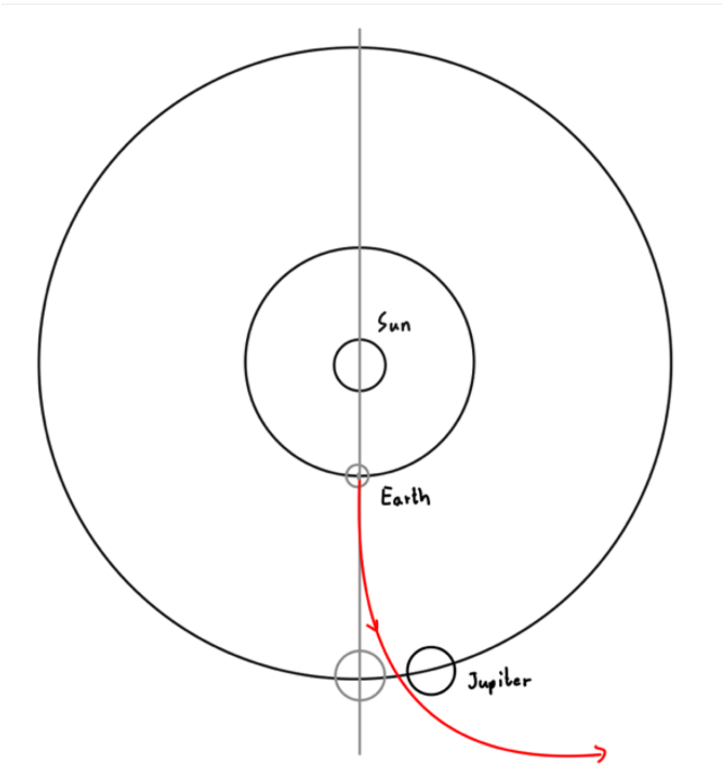


Figure 10: Trajectory of the Earth during the gravitational slingshot

To simplify matters, let us first consider a one-dimensional case in an inertial frame where the velocities of the Earth and Jupiter all lie in one direction (Figure 11). V_E and V_J are the initial velocity of Earth and Jupiter respectively, and V_{E_f} is the final velocity of Earth.

(c) By considering energy and momentum conservation, show that $V_{E_f} \approx 2V_J + V_E$. **[2]**

Solution:

Using conservation of momentum:

$$M_J V_J - M_E V_E = M_J V_{J_f} + M_E V_{E_f} \ \textbf{[0.5]}$$

Using conservation of energy,

$$\frac{1}{2}M_E {V_E}^2 + \frac{1}{2}M_J {V_J}^2 = \frac{1}{2}M_E {V_{E_f}}^2 + \frac{1}{2}M_J {V_{J_f}}^2 \ \textbf{[0.5]}$$

(Note: credit was given for stating that the relative velocity before and after the collision is conserved in an elastic collision.)

Solving for V_{J_f} by with the two equations,

$$V_{E_f} = \frac{2V_J + (1 - \frac{m}{M}V_E}{1 + \frac{m}{M}} \ \textbf{[1]}$$

Since the mass of Jupiter M » mass of Earth m, $\frac{m}{M} \approx 0$. Thus,

$$V_{E_f} \approx 2V_J + V_E$$

Full credit was also given if participants use the change of reference frame method.

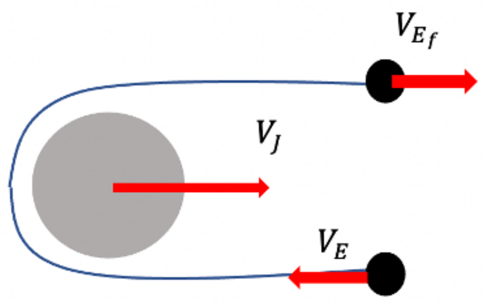


Figure 11: One-dimensional model of a gravitational slingshot

We can now consider the 2-dimensional case in an inertial frame where the Earth approaches Jupiter at an angle θ relative to Jupiter's direction of motion, and is deflected at the same angle, resulting in a symmetric trajectory (Figure 12).

(d) Given that $V_{E_f} \approx 2V_J + V_E$ in the one-dimensional case, derive the expression for V_{E_f} for the two-dimensional case in terms of θ, M_J and M_E. **[3]**

Solution:

$$\textbf{X-axis: } V_{E_x} = V_E \cos\theta \ \textbf{[1]}$$

$$\textbf{Y-axis: } V_{E_y} = V_E \sin\theta \ \textbf{[1]}$$

The x-component velocity of the equation changes according to the relation $V_{E_{x_f}} \approx 2V_J + V_{E_x}$. However, the y-component of the velocity remains unchanged.

$$V_{E_f} = \sqrt{(2V_J + V_E cos\theta)^2 + (V_E sin\theta)^2}$$

$$= \sqrt{4{V_J}^2 + {V_E}^2 + 4V_E V_J cos\theta} \quad \textbf{[1]}$$

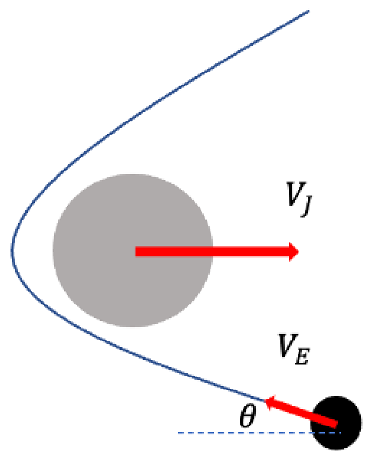


Figure 12: Two-dimensional model of a gravitational slingshot

This relatively simple derivation should gives us some intuition as to why the Earth can gain a velocity boost from slingshotting around Jupiter. However, notice that in our derivation, even though we have specified angle at which the earth approaches Jupiter, θ, this does not actually specify the trajectory of the earth (imagine displacing the earth while leaving its velocity direction unchanged; logically it should result in a different trajectory, but our derivations do not take that into account). In fact, by imposing the symmetry requirement, we have already made a particular implicit choice of trajectory. To properly describe the trajectory of Earth, we need a few more ingredients.

Part III Deflection of Earth's Trajectory

Let us assume a simplified gravitational slingshot model where the larger mass M is far greater than the smaller mass m ($M >> m$). We will also assume that the smaller mass m is approaching a stationary M from a very great distance away (this means that mathematically we can treat m as having an initial position at infinity) with an initial velocity v_0. The trajectory of the smaller mass m will be deflected by the gravitational force of the larger mass M (see figure 13) and approach a direction at an angle ϕ relative to its original direction of travel. ϕ is thus also called the angle of deflection.

Since we are assuming that m is negligible relative to M, we can assume that M is practically stationary throughout the interaction (you will justify this assumption later on), and thus by conservation of energy, the smaller mass m approaches a final velocity equal to v_0.

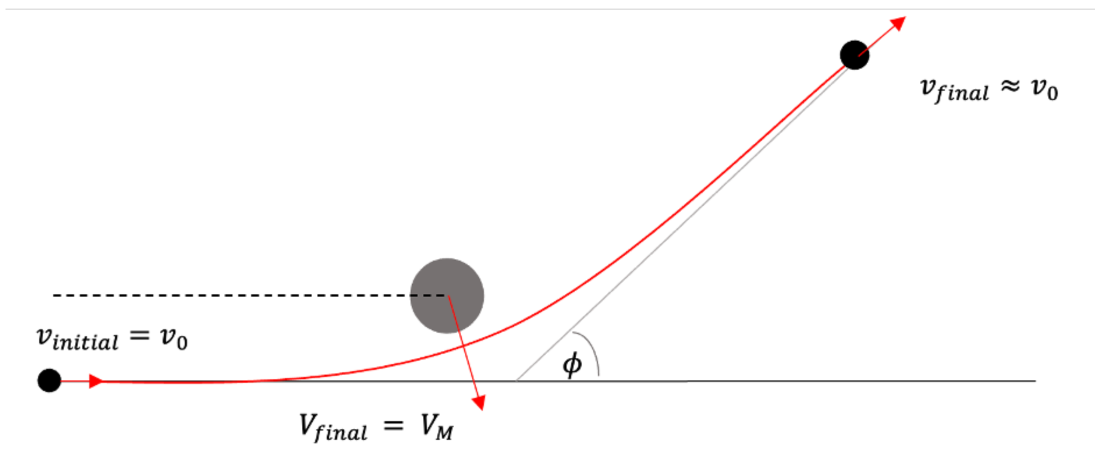


Figure 13: The deflection by a moving mass by the gravitational force with an angle of deflection ϕ.

Also, since we assumed that the larger mass M is stationary, we can treat its position as constant, and treat it as a convenient origin for a set of polar coordinates (r, θ). The trajectory of the smaller mass can thus be described by the following equation:

$$r = \frac{C}{1 + e \cdot cos\theta} \tag{5}$$

where C is a constant, e is the eccentricity of the trajectory, θ is the angle measured from the line of symmetry of the trajectory. We will also further define a quantity called the impact parameter, b, which is the distance between M and m *perpendicular* to the direction of the initial velocity v_0 (see figure 14). As we will see, b is the sole free parameter that determines the angle that m is deflected by M.

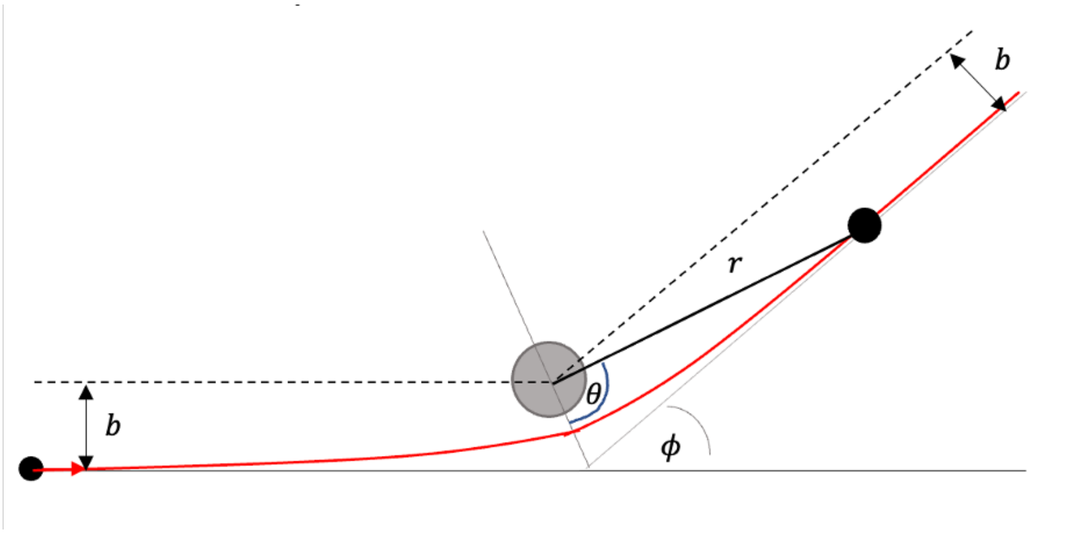


Figure 14: The trajectory of the smaller mass as described by the polar coordinate equation.

In the case of $M >> m$, C and e can be written as:

$$C = \frac{L^2}{GMm^2} \qquad e = \sqrt{1 + \frac{2E}{m}(\frac{L}{GMm})^2} \tag{6}$$

where E and L is the total energy and angular momentum of the system respectively.

Gravity is a central force, meaning it only ever acts radially along the same line joining two masses. For a collection of masses moving only under the influence of central forces, its total angular momentum L will always be conserved. In 2 dimensions, we can define L relative to some origin by:

$$L = mr^2\omega \tag{7}$$

where r is the distance of the mass m from the origin, and $\omega = \frac{d\theta}{dt}$ is its angular velocity relative to the origin. Since angular momentum is conserved, we can greatly simplify our equations by considering the original angular momentum the small mass m has at infinity. Similarly, since total mechanical energy is conserved, we can consider the total mechanical energy possessed by the mass m at infinity.

(e) Express the angular momentum L and total energy E of the small mass m in terms of the initial speed of the small body v_0 and the impact parameter b. **[2]**

Solution:

Angular momentum with respect to the larger mass M:

$$L = mv_0b \text{ [1]}$$

Total energy:

$$E = \frac{1}{2}m{v_0}^2 - \frac{GMm}{r}$$

At infinity, $r \to \infty$, thus the gravitational potential $\frac{GMm}{r} \to 0$.

$$E = \frac{1}{2}m{v_0}^2 \text{ [1]}$$

(f) Thus, express the constant C and eccentricity e as given by (6) in terms of the mass of the large body M, v_0 and b. **[1]**

Solution:

Plugging the expressions for L and E into C and e,

$$C = \frac{{v_0}^2 b^2}{GM}$$

$$\mathbf{e} = \sqrt{1 + \frac{b^2 {v_0}^4}{G^2 M^2}}$$

(g) From equations (5) and (6), derive the expression for r_m, the closest distance between mass m and mass M as mass m moves along its trajectory, in terms of M, v_0, and b. **[1]**

Solution:

The closest distance is on the symmetry line, which is at $\theta = 0$.

$$r = \frac{C}{1+e} \text{ [0.5]}$$

$$\mathbf{r} = \frac{\frac{{v_0}^2 b^2}{GM}}{1 + \sqrt{1 + \frac{b^2 {v_0}^4}{G^2 M^2}}} \text{ [0.5]}$$

Full credit is also awarded if conservation of energy and angular momentum are used to solve for the closest distance.

By considering the equation of the trajectory (5) , we find that θ approaches a finite value θ_{max} as r approaches infinity. θ_{max} is related to the deflection angle ϕ by the following geometric relation (which can be also derived from the diagram):

$$\phi = 2\theta_{max} - \pi \tag{8}$$

(h) Thus, derive the expression for the deflection angle ϕ in terms of v_0, b, and M. **[2]**

Solution:

Rewriting the polar equation,

$$\frac{1}{r} = \frac{1}{C} + \frac{e\cos\theta}{C}$$

At very far positions $r \to \infty$,

$$0 = \frac{1}{C} + \frac{e\cos\theta}{C}$$

Since $\cos(-\theta) = \cos\theta$**,**

$$\theta_max = \pm\cos^{-1}(\frac{1}{e}) \ [\mathbf{1}]$$

From geometry,

$$\phi = 2\theta_{max} - \pi$$

$$\phi = 2\cos^{-1}(\frac{1}{e}) - \pi \ [\mathbf{1}]$$

$$\phi = 2\cos^{-1}(\frac{1}{\sqrt{1+\frac{b^2{v_0}^4}{G^2M^2}}}) - \pi$$

To simplify our calculations, we have assumed that the larger mass M is stationary before and after the interaction. In reality, due to conservation of linear momentum, the larger mass will have a nonzero final velocity V_M after the interaction. We have to show that our assumption of a stationary M is a valid approximation for $M >> m$.

(i) Show that the final velocity of the larger mass can be written as:

$$V_M = v_0\frac{m}{M}\sqrt{2(1-\cos\phi)}$$

[**2**]

Solution:

Let δ **be the angle of** V_M **with respect to the x-axis. Using the conservation of momentum, the equations in the x and y directions are:**

$$mu = mu\cos\phi + MV_M\cos\delta \ [\mathbf{0.5}]$$

$$0 = mu\sin\phi - MV_M\sin\delta \ [\mathbf{0.5}]$$

Squaring both equations and summing them together,

$$M^2V_M^2(\cos^2\delta + sin^2\delta) = m^2u^2(1-\cos\phi)^2 + m^2u^2\sin^2\phi$$

$$V_M = u\frac{m}{M}\sqrt{2(1-\cos\phi)} \ [\mathbf{1}]$$

From equation (i), we see that in the case of $\frac{m}{M} << 1$, $V_M \approx 0$. This justifies our use of the conservation of energy to approximate the final velocity of the smaller mass m.

In summary, the trajectory (and thus the angle of deflection) of a small mass (like Earth) around a much larger body (like Jupiter) can be completely specified by the impact parameter b. We can of course relate the velocity boost equation we found earlier in part II to the impact parameter b. To do that, we would need to find an equation relating the angle of approach of Earth in the inertial frame to the impact parameter in Jupiter's reference frame. This is left as an exercise to the reader.

Part IV Habitable Zone

After the Earth is sent out of the solar system on a slingshot trajectory by Saitama, it eventually found itself pulled into a circular orbit around a new star weighing 1.5 solar masses and with a temperature 1.2 times that of the sun, with a radius 1.3 times that of the sun, at a new orbital velocity of 30 km/s. The fate of the Earth now depends on whether it had fortuitously entered the habitable zone of its new star.

A hot body with temperature T emits energy per unit second according to the Stefan-Boltzmann equation.

$$P = Ae\sigma T^4 \tag{9}$$

Where P is the radiated energy per unit second A is the surface area of the body e is the emissivity σ is the Stefan-Boltzmann constant T is the temperature of the blackbody

(j) Given that the sun can be modeled as a perfect black body ($e = 1$), with a temperature of 5800 K and radius of 696,340 km, estimate the equilibrium temperature of Earth in its original orbit around the sun (assume a circular orbit and ignore the effects of an atmosphere). [2]

Solution:

The intensity of solar radiation at a distance of a_E away from the sun is:

$$I = \frac{P}{4\pi {a_E}^2}$$

By the Stefan-Boltzmann Law,

$$P = 4\pi {r_s}^2 \sigma {T_s}^4$$

where r_s amd T_s are the radius and temperature of the Sun respectively. Thus,

$$I = \frac{4\pi {r_s}^2 \sigma {T_s}^4}{4\pi {a_E}^2}$$

Assuming that Earth is a perfect blackbody, power received by Earth is calculated by multiplying intensity with the area of the Earth's disk.

$$P_{received} = \frac{4\pi {r_s}^2 \sigma {T_s}^4}{4\pi {a_E}^2} \pi r_E^2$$

At thermal equilibrium, the power received by the earth is equal to the power emitted. The power emitted is given by the Stefan-Boltzmann law with temperature at the equilibrium temperature T_E.

$$e\pi r_E^2 \sigma {T_E}^4 = \frac{4\pi {r_s}^2 \sigma {T_s}^4}{4\pi {a_E}^2} \pi r_E^2 \ [\mathbf{1}]$$

Solving for T_E,

$$T_E = \left[\frac{r_s^2 T_s^4}{4r^2}\right]^{1/4}$$

$$= \left(\frac{696340000^2 \times 5800^4}{4(149.6 \times 10^9)^2}\right) = 280K \ [\mathbf{1}]$$

(k) Estimate the equilibrium temperature of the Earth in its new orbit around the new star. Does the new orbit of the Earth lie in the habitable zone of the star? [2]

Solution:

As derived in part (a), the orbital radius is given by:

$$r = \frac{GM}{v^2}$$

$$r = \frac{6.67 \times 10^{-11} \times 1.5 \times 1.99 \times 10^30}{(30 \times 10^3)^2}$$

$$r = 2.21 \times 10^8 km \ [\mathbf{0.5}]$$

Using the formula for equilibrium temperature derived in part i),

$$T_E = \left(\frac{(1.2 \times 696340000)^2 (5800 \times 1.3)^4}{4(2.21 \times 10^{11})^2}\right)^{1/4}$$

$$= 328K$$

This temperature corresponds to roughly $55\deg C$ where water can still exist as a liquid. Hence, the Earth lies in the habitable zone of the star.

After all his efforts, Saitama is satisfied that he has saved the people of Japan from the vicious summer heat (as for the freezing cold of interstellar space, they're on their own). The only thing that he regrets is that he can no longer visit Earth anymore, as thanks to the conservation of momentum, Saitama found himself ejected out of the solar system in the opposite direction at relativistic speeds.

Question 4 An Afternoon at the Sundial Garden

Part I How does a sundial work, anyway?

One fine afternoon, you decide to take a walk around the Singapore Botanic Gardens to let off some steam and explore the wonders that glorious Mother Nature (read: NParks) has to offer. As you wander about, you stumble into a small clearing surrounded by four small ponds and a grove of trees. In the centre of the clearing, you find a peculiar device mounted on top of the pedestal (Figure 15).

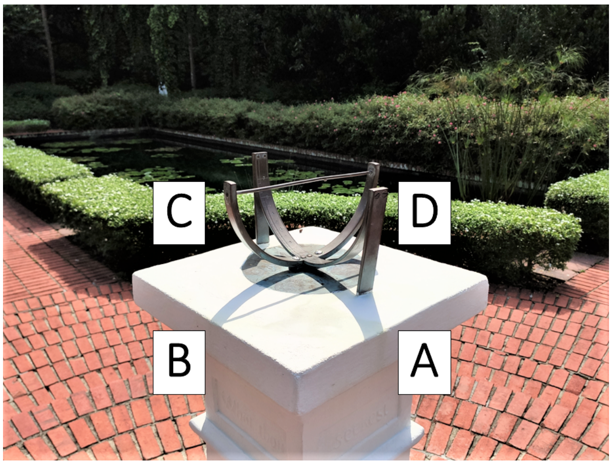


Figure 15: A peculiar device mounted on pedestal

Upon closer investigation, you find a row of hour markings etched into the metal arc. This device must be a sundial, though it certainly did not fit what you thought a sundial would look like. This particular sundial is actually an equatorial sundial with a dial ring.

(a) For each of the directions A, B, C and D labeled in Figure 15, identify the corresponding cardinal direction (North, South, East and West). Note that there can be more than 1 set of correct answers. [1]

Solution:

A: West B: North C: East D: South

or

A: East B: South C: West D: North

Note that the gnomon has to be aligned along the local meridian.

Sometimes you may see an equatorial sundial with a dial plate instead of a ring (Figure 16), with hour line markings on both sides of the dial plate. This is needed because the declination of the sun changes over a year from 23.5°N to 23.5°S, thus each face of the dial plate is sunlit for only half the year.

Figure 16: A sundial with an equatorial plate in the Forbidden City, Beijing[1].

(b) **Does the changing declination of the sun affect the accuracy of time readings[2] on an equatorial sundial? Briefly explain why or why not.** [2]

Solution:

No it does not. [1]

The sun remains at the same hour angle. A change in declination only results in a longer or shorter gnomon shadow. The shadow is longer at solstices and shorter near equinoxes. [1]

(c) **Instead of an equatorial sundial, can we conceivably create an 'ecliptic sundial' (where the dial plate/ring is aligned to the ecliptic plane)? If yes, briefly explain how you would construct one. If no, briefly explain why not.** [2]

Solution:

No, it would be meaningless to try to build one.

In order for the dial plate/ring to be aligned with the ecliptic plane at all times, it will have to be rotated about the axis of Earth's rotation at a constant rate of one full rotation per day. [1]

As the dial plate/ring rotates along with the earth, the shadow of the gnomon will remain on a fixed marking on the dial plate/ring from morning to dusk. [1]

A Sundial Primer

A sundial consists mainly of two parts; the *gnomon* and the *dial*. The dial is usually a surface which may be flat or curved like a ring or globe, with hour line markings engraved. The gnomon is the part of a sundial that casts a shadow on the dial surface, and the position/angle of the gnomon shadow on the dial plate determines the time to be read off.

The most common type of sundial in popular imagination is the London-type dial (Figure 17), which has a horizontal dial plate and a wedge-shaped gnomon. The edge of the gnomon (known as the *style*) is aligned with the axis of rotation of the celestial sphere. Therefore, London-type dials (in fact, sundials in general) are specifically made for a particular latitude and will not provide accurate time readings at other latitudes.

[1] Image by Sputnikcccp. From Wikimedia Commons.

[2] For clarity, this is referring exclusively to apparent solar time

An equatorial sundial in contrast has hour markings engraved on a plate or ring mounted in the same plane as the celestial equator, normal to the gnomon. As a result, and unlike a London-type dial, the hour markings on an equatorial dial are equally spaced apart.

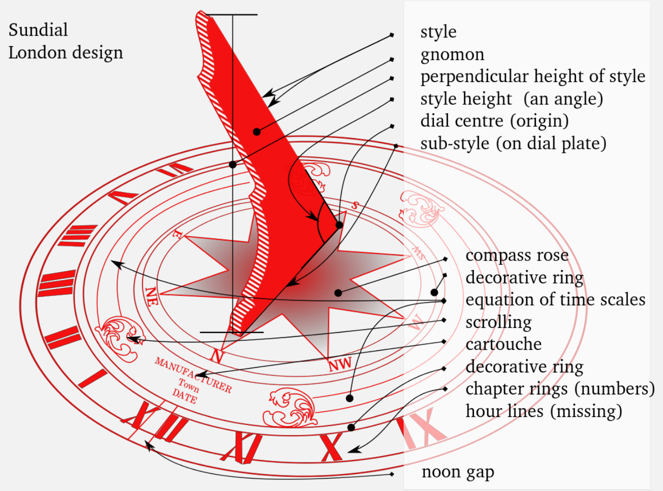


Figure 17: Anatomy of a London-type dial [a]

[a] Image by Clem Rutter, Rochester, Kent. From Wikimedia Commons.

Part II Nodus and Gnomon

Inspired by your recent pilgrimage to the Sundial Garden, you decide to construct a sundial of your own based off vague memories of a sundial-making activity in Kindergarten. To construct the sundial, you push a pencil through a hole in a paper plate. The pencil thus acts as a vertical gnomon, and empirical hour lines can be drawn on the paper plate by tracing the gnomon shadow at a given watch timing.

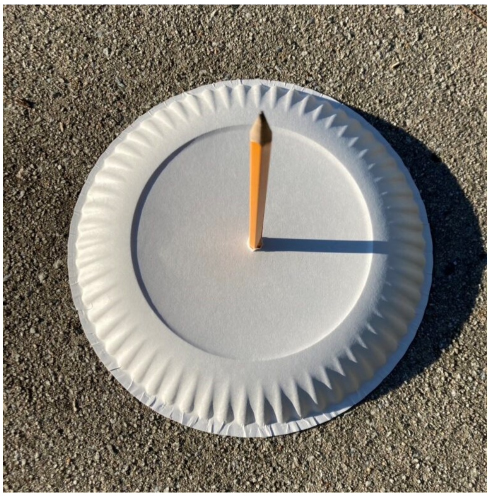


Figure 18: A rudimentary sundial [3]. Empirical hour lines are made by tracing the gnomon shadow at regular time intervals

Wait, something's not right, you thought. A gnomon should be aligned with the axis of the Earth's rotation, and at equatorial latitudes, it would be mean that your gnomon needs to be lying flat on the ground! So what exactly have you created? Is this even a sundial? What other lies were you told back in Kindergarten?

What you have made is closer to what we may call a *nodus-based sundial.* Unlike equatorial and horizontal sundials with an axial gnomon, the gnomon shadow of a nodus-based sundial may not be aligned with fixed hour lines. Time instead is determined by the shadow position of a designated point on the gnomon, called the nodus. In the above example, we can treat the very tip of the pencil shadow as the nodus, and use its position on the hour line markings to tell the time accordingly.

(d) For the sundial with a vertical gnomon shown in Figure 18, state what physical quantity do the 'hour lines' actually measure and explain why does the shadow of a vertical gnomon not correspond to an hour line in general. [1]

Solution:

The 'hour lines' measure the azimuthal angle of the sun. [0.5] Two celestial objects at the same hour angle but different declinations will have different azimuthal bearings. [0.5]

(e) Therefore, suggest how should the hour line markings be drawn for such a rudimentary sundial at a latitude of 0°N. [1]

Solution:

The hour line markings have to be drawn parallel to the meridian in the North-South direction.

[3] Sarah McClelland. (2021, July 29). *How to make a sundial.* https://littlebinsforlittlehands.com/how-to-make-a-sundial/.

In general, the hour lines for both nodus-based and axial gnomon sundials are projections of hour circles onto the plane of the gnomon's shadow. The difference is that for a sundial with an axial gnomon, the base of the gnomon style is guaranteed to be the intersection of hour circle projections on the dial plate. In other words, hour line markings can be defined by an angle made with the noon line, known as the hour line angle (not to be confused with the hour angle). For nodus-based sundials, the hour lines will eventually converge at a point away from the base of the gnomon.

(f) In practice, a London-type horizontal sundial with an axial gnomon will generally offer a greater range of clock time readings than a nodus-based horizontal sundial. Briefly explain why. **[1]**

Solution:

Unlike a London-type sundial, a nodus-based sundial won't be able to show times near sunrise or sunset, when the sun is low in the sky. [0.5]

As the sun approaches the horizon, the shadow length of the gnomon (and thus the position of the nodus) stretches to infinity, and thus the range of time that can be shown by a nodus-based sundial is severely limited by the size of its dial plate. However, since a London-type sundial shows time by the angle of the gnomon shadow, the shadow length does not affect the range of time that can be shown. [0.5]

By considering the geometry of the gnomon, the gnomon shadow and the direction of the sun's rays (Figure 19), we can derive an equation relating the hour line angle θ, the hour angle of the sun H and the local latitude ϕ. This equation is given by $\tan\theta = \sin\phi \times \tan H$.

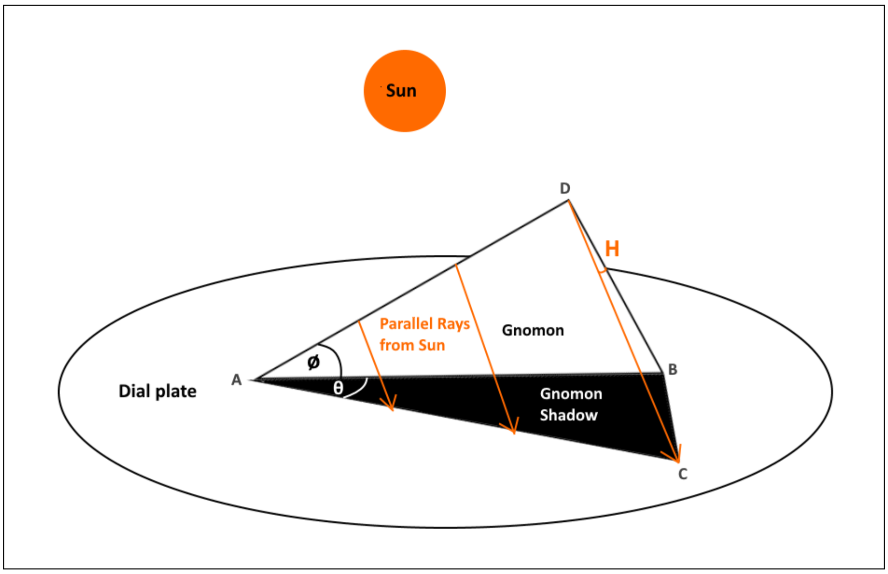


Figure 19: To simplify the setup, the sun is fixed at zero declination, and thus angles ADB, DBC and ABC are all right angles. This allows us to derive the hour line angle equation by considering simple trigonometric relations between the sides.

(g) Based on the given formula or otherwise, explain why a London-type dial may not be particularly practical at latitudes close to (but not on) the equator. **[2]**

Solution:

At near-equatorial latitudes, the hour lines representing hours near noon will be clustered very closely around the noon line, making it impractical to determine the exact time reading. [1]

We can deduce this by various methods: [1]

1) We can analyse the function by considering how θ varies as a function of H, while treating $\sin\phi$ as a very small constant multiplicative coefficient. At small H and θ, the tangent function is roughly linear, and a large variation in H leads to a small variation θ due to the small coefficient. This means that the hour lines are closely packed together. However, as H approaches $\frac{\pi}{2}$, $\tan H$ approaches infinity and the constant multiplicative coefficient becomes insignificant, and θ starts to scale proportionally with H, resulting in spaced-out hour lines.

2) By tabulating real values of θ and H we see that the hour line angles are very close to each other for H close to 0 and further apart for H close to $\frac{\pi}{2}$.

Hour Circles and Hour Angles

Hour circles are longitudinal circles which intersect at the north and south celestial poles and are similar to Right-Ascension lines in a RA-Dec coordinate system. The hour angle of an object is defined by the angle between the plane of its hour circle and the plane of the local meridian (which is also the noon hour circle). The hour angle of the sun, in essence, defines the local solar time at a particular location.

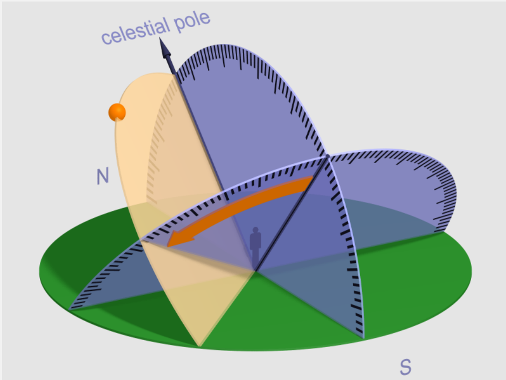


Figure 20: Diagram illustrating how the hour angle is defined[a].

[a]Image by Sch. From Wikimedia Commons.

Part III Time Axes

Over the course of a day, the shadow of a sundial nodus will trace out a curve as the sun rises and sets. However, over a year as the declination of the sun changes, the height and position of the nodus will change for a given time of the day, resulting in a different curve traced out. These curves are therefore called the *declination lines* of the sundial. Declination lines are non-intersecting and are part of a family of mathematical curves known as conic sections. At mid-latitudes, the set of declination lines form a shape resembling a double-bladed axe (Figure 21).

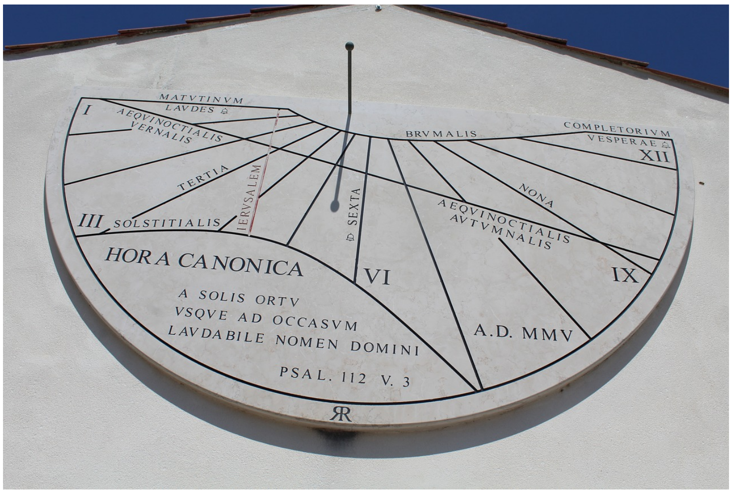


Figure 21: A vertical nodus-based sundial with declination lines inscribed. Note that the hour lines are not symmetrical about the noon line as the wall is not aligned in the E-W direction.

As each declination line corresponds to a particular solar declination, the position of the nodus shadow can be used to tell the time of the year, with some caveats.

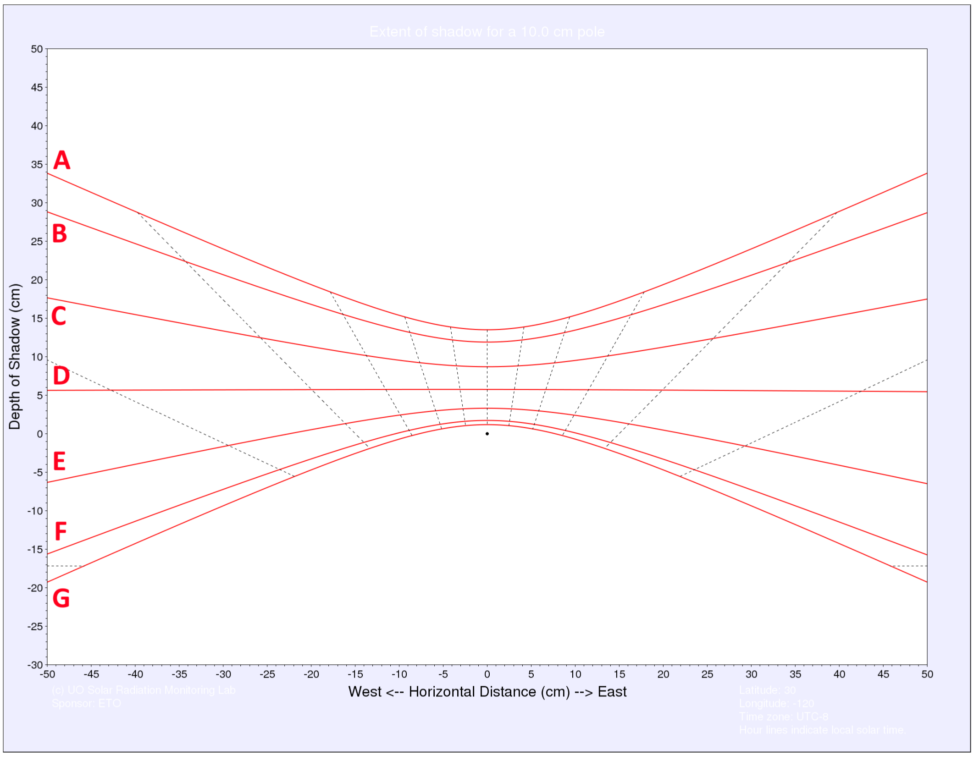


Figure 22: Declination lines generated for a location at 30°N

(h) **Figure 22 shows a set of declination lines for a location at a latitude of 30°N. State which of the 7 declination lines (labeled A to G from the top) does nodus shadow fall on for the following dates:** **[2]**

- **(i) Spring Equinox**
- **(ii) Summer Solstice**
- **(iii) Autumn Equinox**
- **(iv) Winter Solstice**

Solution:

(i) Spring Equinox: D [0.5]

(ii) Summer Solstice: G [0.5]

(iii) Autumn Equinox: D [0.5]

(iv) Winter Solstice: A [0.5]

At equinoxes, the sun lies on the celestial equator and thus its diurnal circle is a great circle that contains the observer (sundial nodus) at its origin. Therefore the rays of the sun intersecting the sundial nodus sweep out a flat plane that intersects with the ground along a straight line.

At summer solstice, the gnomon's shadow is at its shortest since the sun is at its highest point in the sky. Vice versa for winter solstice.

(i) **Therefore explain why it may be difficult to determine the exact date marked by the position of the nodus shadow on the declination lines on the days near solstices.** **[2]**

Solution:

The declination of the sun is roughly the same on dates that are the same number of days before and after solstices, since the direction of the change in declination of the sun flips after solstice. [1]

Therefore near solstices, dates that are only a few days apart can lie on the exact same declination line, making it difficult to tell the exact date. [1]

Note: 1 mark was given for explanations based on the close clustering of declination lines on days near solstices.

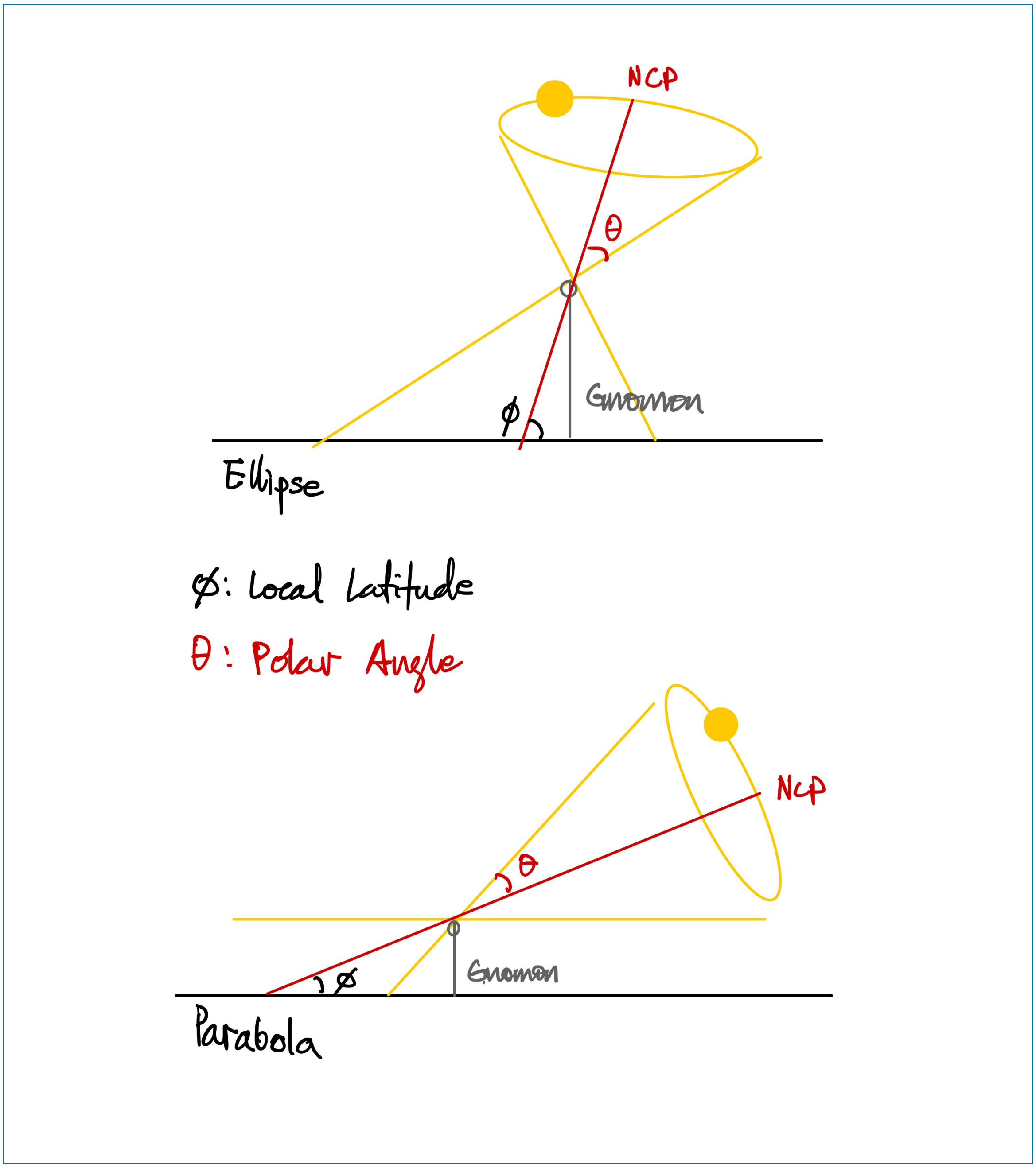
The declination lines shown in Figures 21 and 22 are generally hyperbolic curves, which are a type of conic section. The reason that hyperbolic curves are formed is because over the course of a day, the sun traces out a circular path in the sky along lines of constant declination, called diurnal circle. Light rays from the sun passing through a sundial nodus will thus trace out a conical surface over the course of a day, and the intersection of the ground with this cone produces a conic section.

(j) Is it possible for declination lines to be parabolas or ellipses? If so, explain what are the conditions for such declination lines to occur. If not, explain why. You may sketch a diagram to supplement your explanation if you wish. [2]

Solution:

Yes, this is possible at latitudes above the arctic circle ($66.5°$N). Elliptical declination lines occur when the sun is circumpolar (midnight sun). [1]

Parabolic declination lines occur when the sun is circumpolar and the sun is exactly on the horizon at its lower culmination. This happens when the local latitude is equal to the polar angle ($90°-$ declination), which can only happen above the arctic circle. [1]



Conic Sections

Circles, Ellipses, Parabolas and Hyperbolas are all part of a family of curves called conic sections. Conic sections are named thus because they can be constructed geometrically by intersecting a flat plane with a cone, and the type of curve produced depends on the relative angle between the flat plane and the slope of the cone (see Figure 23).

A circle is formed when the intersecting plane is exactly normal to the cone's axis of rotation, and a parabola is formed when the intersecting plane is exactly parallel to to the slope of the cone. An ellipse is formed when the intersecting angle is between that of a circle and a parabola, and a hyperbola is formed when the intersecting angle is greater than that of a parabola.

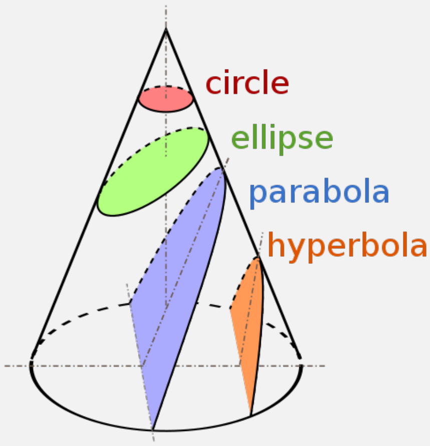


Figure 23: Conic Sections

Part IV The Equation of Time

In many sundials, you will find a small curve graph known as the *equation of time*, which tells you how many minutes have to added or subtracted from the time shown on a sundial to obtain the actual watch time on any particular day of the year. A full-size equation of time graph is attached in the Appendix for your reference.

(k) Let us consider the sundial in figure 18 with empirical hour lines marked according to a watch showing Standard Singapore Time (SST) on 1st of October. If the sundial shows that the time is 12:00pm on 1st of March, what is the actual watch time in SST? [2]

Solution:

Firstly, the time standard used (SST) is irrelevant since all time standards are based off the mean solar time at some reference meridian, and thus will be offset by a constant time interval when compared to the local mean solar time.

Next, we note that the y-axis of the equation of time chart states Apparent Solar Time - Mean Time. Therefore, the offset has to be deducted from the apparent solar time read off the sundial to obtain mean solar time. Conversely, the offset has to be added to mean solar time to obtain the corresponding apparent solar time.

By referencing the equation of time chart, we find that the offset on 1st October is +10 minutes and -12.5 minutes on 1st March. Difference in time offset between the two dates is 22.5 minutes. Therefore the actual time in SST is 11:37.5 am. [2]

It should be possible to construct the hour lines on a nodus-based sundial such that it automatically compensates for the equation of time. This is done with a curved hour line that is ahead or behind the original straight hour line at points corresponding to different solar declinations, with the time difference given by the equation of time.

(l) Describe the shape of the hour lines if such a correction was to be applied. You may supplement your description with a sketch if needed. [2]

Solution:

The shape will be a figure of eight. In fact, it will be in the shape of the solar analemma (though vertically and laterally inverted as well as skewed depending on the angle of the shadow plane). [2]

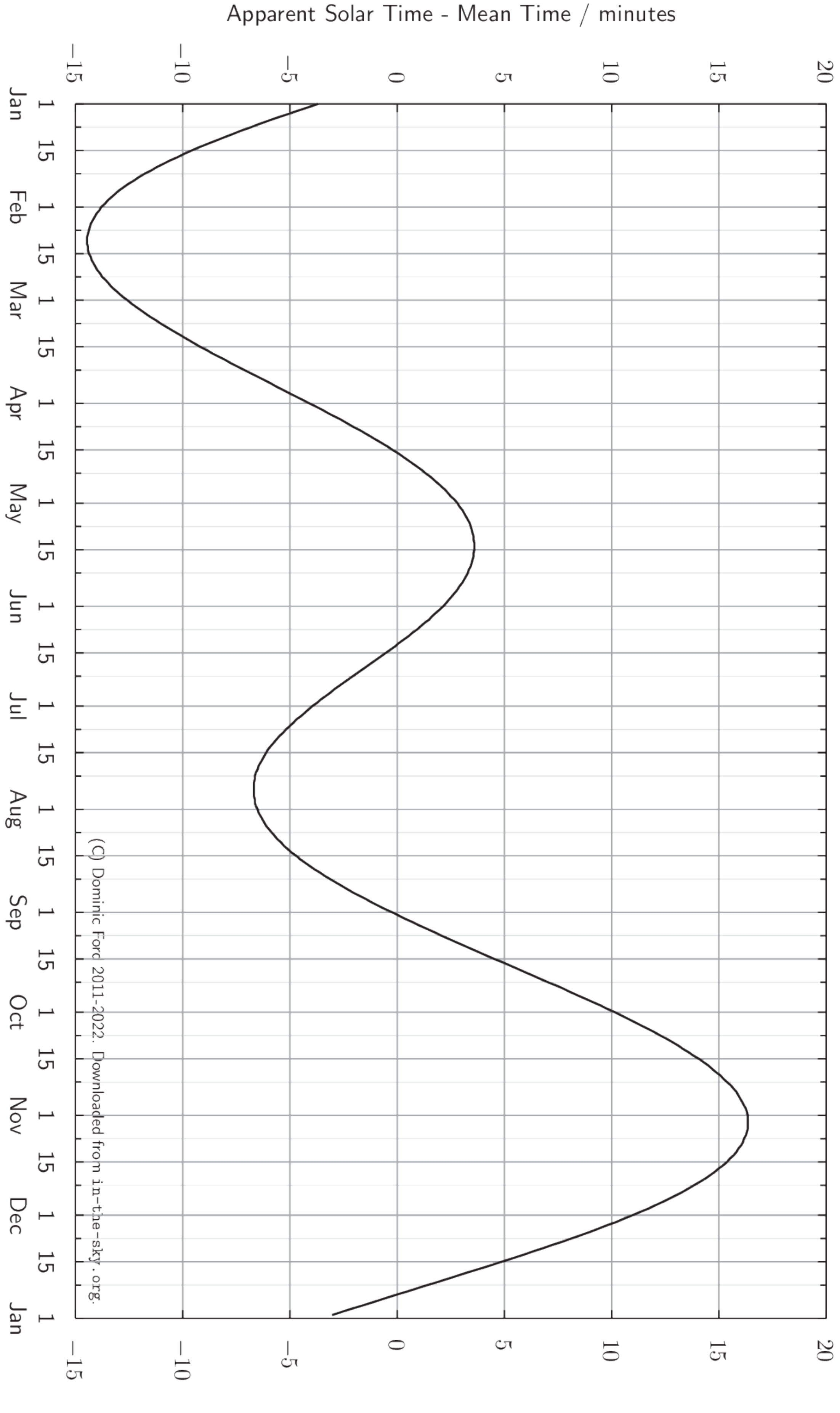
This is because the equation of time is simply the East-West component of the solar analemma, but with the x-axis parameterized by time instead of declination angle. To put it in another perspective, the position of the sun at the same mean time will trace out the analemma over the year, and similarly the shadow of the nodus will trace out the analemma shape as well. This analemma shadow is exactly our corrected hour line.

Clock Time and Sundial Time

Due to the obliquity of the ecliptic and the eccentricity of Earth's orbit around the Sun, the Right Ascension of the Sun does not increase at a constant rate throughout the year, and therefore the time between successive local noons varies. If we imagine a *mean sun* that moves along the ecliptic with a constant increasing Right Ascension, the time between successive local noons is fixed. Over the course of a year, the actual position of the Sun (called the *apparent sun*) will fall behind or move ahead of the Mean Sun at different parts of the year. A sundial measures time according to the actual position of the sun in the sky, and therefore shows what is known as *apparent solar time*. A clock on the other hand measures time according to a constant reference oscillation, and therefore shows *mean solar time*. This difference between the apparent and mean solar times over a year is captured by the equation of time.

Most modern clocks however are set according to a time standard such as GMT, which is the Mean Solar Time at Greenwich, London. If we build a sundial and apply the equation of time to obtain the (local) mean solar time, we will find that it diverges from clock time as defined by time zone standards.

Part V Appendix



 [End of paper]