AstroChallenge 2018
Data Response Questions
(Senior)

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. This paper consists of **21** printed pages and **4** blank pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your school (and team number) on **EVERY** page, as well as the individuals in the said team on the first page.
6. It is your team's responsibility to ensure that all pages of your answer script have been submitted, **including pages to be detached from this booklet**.

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Question 1
Getting Crabby
(20 points)

Born from a supernova in 1054, the Crab Nebula has the distinction of being the only supernova remnant on the Messier Catalogue. Today, the Crab Nebula is known to be one of the harder winter Messier objects to observe: small scopes typically only reveal a dim glow even in good skies. Yet, plainly it was bright enough in the past for Charles Messier to mistake as a comet despite his mediocre telescope (by modern standards). This leads us to the question: how much easier was the Crab Nebula to observe in the past? Let us rewind the clock...

The table below summarises some of the current known information about the Crab pulsar.

Characteristic	**Symbol**	**Value**
Apparent magnitude	$m(V)$	$+16.5$
Distance	d	$1.9\,\text{kpc}$
Mass	M	$1.4M_\odot$
Radius	R	$10\,\text{km}$
Rotational period	P	$33\,\text{ms}$
Temperature	T	$1.6 \times 10^6\,\text{K}$

Table 1: Some characteristics of the Crab pulsar.

(i) [1 point] What is the current luminosity of the Crab pulsar solely due to black-body radiation L_B?

For comparison, the entire Crab Nebula has been determined to have a current luminosity L of $5 \times 10^{31}\,\text{W}$ (Kennel and Coroniti, 1984). The Crab Nebula would have quickly cooled to background temperatures if the Crab Pulsar's black-body radiation was the sole source of thermal energy.

An alternative source of energy is rotational magnetic braking. We know that pulsars are rapidly rotating and have strong magnetic fields. As magnetic field lines move past the surrounding material, the surrounding material exerts a force that opposes this motion (by Lenz's law). The net result is that the pulsar loses rotational kinetic energy to its surroundings, in a process known as spin-down.

Suppose we have an object spinning with period P. It is known that the rotational kinetic energy of an object is related to its angular velocity ω by the relation

$$E_{\text{rot}} = \frac{1}{2} I\omega^2,$$

where I is the moment of inertia. For the case of a perfectly uniform sphere, $I = \frac{2}{5}MR^2$, where M and R are the mass and radius of the sphere respectively.

(ii) [2 points] Assume that the Crab Pulsar is perfectly spherical with a constant moment of inertia. Show that the amount of energy emitted per second due to magnetic braking L_{spin} is given by

$$L_{\text{spin}} = 4\pi^2 I P^{-3} \frac{dP}{dt},$$

where $\frac{dP}{dt}$ is the change in the pulsar's period as the pulsar ages.

(Hint: Think of the physical picture. As the pulsar slows, the rotational kinetic energy of the pulsar is being lost to space, which is observed as L_{spin}. This implies that $L_{\text{spin}} = -\frac{d}{dt}[E_{\text{rot}}]$.)

(iii) For the case of the Crab Pulsar, $\frac{dP}{dt} = 4.22 \times 10^{-13}$ (seconds per second).

(a) [1 point] Hence or otherwise, calculate L_{spin}.

(b) [1 point] By comparing this value to L_B and L, suggest if rotational magnetic braking is sufficient to power the Crab Nebula.

Under suitable assumptions, the period of a pulsar and its spin-down rate are related by the formula

$$\tau = \frac{P}{2 \times \frac{dP}{dt}},$$

where τ is the characteristic age of the pulsar.

(iv) [3 points] Assuming that the characteristic age of the Crab pulsar is equal to its age t (i.e. $\tau = t$), prove that the relationship between P and t is given by

$$P = K\sqrt{t},$$

where K is a constant. You **do not** need to compute a value for K. (its numerical value does not need to be found). [3 marks]

We are interested in seeing how the luminosity of the nebula changes with time, so let us figure that out next.

(v) [1 point] Let the luminosity of the nebula at time t be L_t. For simplicity, let us assume for the rest of this question that $L_t = L_{\text{spin}}$.

Express L_t as a function of t, I, K, and other numerical constants only.

(vi) [1 point] Charles Messier first observed the Crab Nebula in 1758. What is the luminosity of the Crab Nebula now compared to then? Express your answer as a percentage.

(vii) [2 points] Let your numerical answer in Part (vi) be Z. Does this mean that the nebula right now (as a whole) is visually Z times as bright as it was in 1758? Why or why not?

This is all well and good, but as seasoned astronomers, you should be familiar that total brightness is not worth much. Rather, in order to gauge visibility, what matters more is surface brightness. To this end, let us now attempt to find out how the average surface brightness of the Crab has changed.

The Crab Nebula itself currently has a radius of 5.5 light years and is expanding at $1500\,\mathrm{km\,s^{-1}}$, and we know the expansion rate is being slowed by gravity. We also know that the entire nebula has a mass of 4.6 solar masses, separate from the pulsar itself. While the nebula is an extended object, given that it is spherically symmetric, we can apply the Shell Theorem. The Shell Theorem allows us to treat the nebula as *decelerating due to a massive "point mass" of mass M located at its centre*. In this case, M refers to the combined mass of the nebula and the central pulsar.

(viii) [1 point] Let the radius of the nebula be R. Consider a test mass expanding with the edge of the nebula (a.k.a. R). Write down the second-order differential equation relating acceleration $\frac{d^2R}{dt^2}$ to M, R, and other constants where appropriate.

(Hint: Analyse the forces involved.)

(xi) [3 points] Find a relationship between the expansion velocity v and R, including any other appropriate constants. Label and explain any constants that you introduce.

(Hint: While it may be tempting to solve the differential equation in Part (viii) to obtain an answer, there is another (and better) way to approach this question.)

It can be shown that if we plug in the given values and adopt certain simplifying assumptions, the expansion velocity of the nebula in the recent past is approximately constant. While the proof of this is left as an exercise for the reader (to be done at the reader's own leisure and certainly not within this DRQ's time limit), an important implication is that the current radius of the nebula R obeys the relationship

$$R \approx X + vt,$$

where X denotes the extrapolated size of the nebula initially.

(x) [1 point] What is the value of X in meters?

(xi) [1 point] Assume that the Crab Nebula is stationary relative to us. Hence or otherwise, what is the angular size of the nebula now compared to 1758? Express your answer as a percentage.

Recall that the apparent surface brightness of an object S has units of magnitude per square arcminute and is given by

$$S = m + 2.5 \lg A,$$

where lg here indicates the base-10 logarithm $\log_{10}$, and A is the apparent size of the nebula. Assume that as in part (vii) that the nebula was visually Z times brighter in 1758 than in 2018.

(xii) [2 points] Calculate the change in average surface brightness since 1758, in terms of magnitude per unit area.

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Question 2
Are Black Holes Really Black?
(20 points)

Part 1: Gravitational Lensing **(Sub-total: 10 points)**

Gravitational lensing is a phenomenon where light from a distant source may be deflected by the curvature of space-time caused by a massive lensing object close to or in the line of sight between an observer and a distant object. We will take the lensing object to be a black hole in this problem. Take a look at the following illustration:

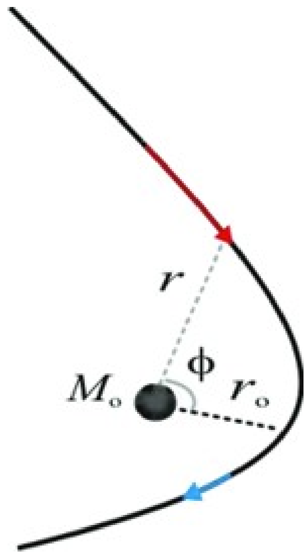

Figure 1: Path taken by light under the Schwarzschild metric. Note that the curvature in this diagram is greatly exaggerated.

The illustration above depicts the path that is taken by light (photon) under Schwarzschild metric, which is a metric that corresponds to the solution of Einstein's field equations with the assumption of chargeless and non-spinning perfectly spherical object and zero cosmological constant. It can be shown that the path can be described by the following equation in polar coordinate:

$$\frac{1}{r} = \frac{\cos\phi}{r_0} - \frac{r_s}{2r_0^2}\cos^2\phi + \frac{r_s}{r_0^2},$$

where r_s is the Schwarzschild radius of the black hole (i.e. the radius of the event horizon of the black hole), r_0 is the minimum distance from the path to the gravitating object, r is the distance from the object to a point along the path and ϕ is the angle between the line connecting the object with a point along the path and the line connecting the object with the point of closest approach.

(i) [2 points] By using Newtonian mechanics and equating the escape velocity at the Schwarzschild radius to the speed of light, derive the expression for the Schwarzschild radius, r_s, as given in the formula book. This derivation is heuristic: the steps are invalid, but it happens to be true. Can you explain why is it invalid?

You may assume that $r_s \ll r_0$ such that the deflection of light from its original path is small.

(ii) [1 point] What is the value of ϕ (in radians) when the photon is very far and the mass of the black hole approaches 0?

(iii) [1 point] Starting from the equation given, show that the path of the photon is a straight line if there is no black hole, as it should be.

(iv) [2 points] Show that the total angular deflection of the photon can be expressed as

$$\delta = \frac{2r_s}{r_0}.$$

You may assume that when x is very small, we have $\sin x \approx \tan x \approx x$ and $\cos x \approx 1$.

(v) [2 points] Suppose we have an event where a black hole passes in front of a star such that the star, the black hole and Earth forms a straight line. This situation is modelled in the diagram below.

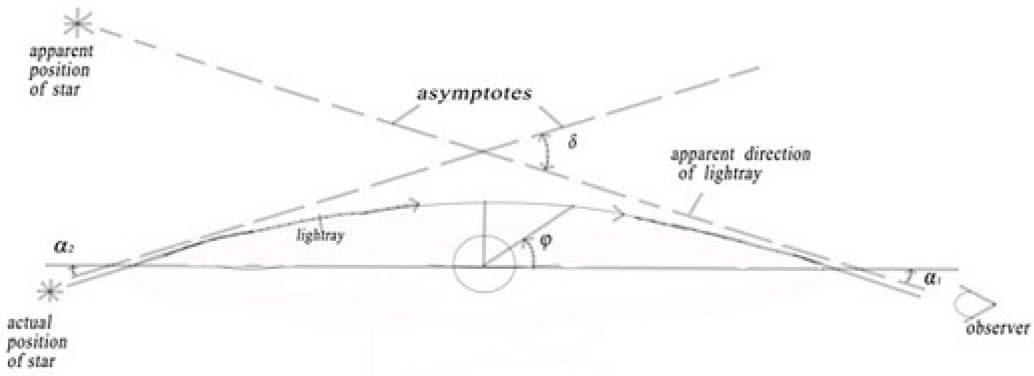

Figure 2: The scenario.

Suppose the black hole is located D_{BE} away from Earth and the distance from the black hole to the star is D_{SB}. Show that the following relation holds:

$$r_s = \frac{1}{2} r_0^2 \left(\frac{1}{D_{\text{BE}}} + \frac{1}{D_{\text{SB}}} \right).$$

(Hint: First, use the small angle approximation $\sin x \approx \tan x \approx x$. Then, note that if the angle is sufficiently small, the intersection point between two dashed lines should be really close to the point of closest approach.)

(vi) [2 points] Suppose we have an event where a black hole located 7.5×10^{20} m away from Earth passes in front of a star 3.0×10^{21} m away from Earth. It is observed that the angular deflection of the star image observed from the Earth (often called the angular Einstein radius, denoted by α_1 in the figure) is 0.025 arcseconds. Calculate the mass of the black hole in terms of solar masses.

Part 2: Hawking Radiation **(Sub-total: 10 points)**

If only classical physics is taken into account, black holes cannot emit radiation and their blackbody temperature can thus be considered to be zero. However, Stephen Hawking showed in his paper in 1974 that when quantum corrections are considered, black holes emit radiation according to the black-body spectrum. The corresponding black-body temperature is known as the Hawking temperature, T_H. For a Schwarzschild black hole of mass M, the Hawking temperature can be written as

$$T_H = \frac{\hbar c^3}{8\pi G k_B M},$$

where $\hbar$ is the reduced Planck's constant, G is the gravitational constant, and k_B is Boltzmann's constant.

According to the second law of thermodynamics, entropy of an isolated system never decreases. The definition of entropy S associated with a physical body is

$$dU = T\,dS,$$

where U is the internal energy of the body and T is the temperature of the body. In the case of a black hole, we may assume that the internal energy is the energy related to its mass (i.e. via Einstein's famous formula), and T is the black-body temperature.

(vii) [2 points] Show that the entropy of a black hole may be expressed as

$$S = KM^2,$$

where K is some constant. Write the value of this constant in terms of $c, G, \hbar$, and k_B.

A merger of two black holes was recently detected by Kip Thorne, Rainer Weiss, and Barry Barish, the winners of the 2017 Nobel Prize in Physics. The first detection of gravitational waves came from the merger of two black holes each with a mass equal to 30 solar masses, located at a distance of 1.3×10^9 ly from Earth.

(viii) [2 points] Assuming that the kinetic energy of the black holes are negligible compared to their rest energy when they collide, show that if two black holes with the same mass collide to make a bigger black hole, at most 30% of their initial rest energy can be converted to gravitational wave radiation.

(Hint: The entropy of an isolated system never decreases.)

(ix) [1 point] Hence or otherwise, give an upper bound for the energy flux expected to pass through a detector situated on Earth in $\mathrm{mJ\,m^{-2}}$.

We will now consider black hole evaporation. Assuming that there is no in-falling matter or energy, a black hole will slowly radiate away its mass through Hawking radiation. Although a correct treatment of the evaporation process at high energy scales requires a theory of quantum gravity, as long as T_H is below the Planck scale, the semi-classical approach we have been using so far suffices. In what follows, we will obtain an estimate of the black hole evaporation timescale; this will be an estimate for the lower bound on the evaporation process duration.

You are given that the value of Stefan-Boltzmann constant is

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}.$$

(x) [1 point] Determine the power radiated by a black hole as a function of mass, assuming that the effective area is the area of the event horizon.

(xi) [2 points] Hence, or otherwise, calculate the evaporation time of a black hole with the same mass as the Sun in seconds. Comment on the result with a suitable comparison.

Next, consider a black hole exposed to the Cosmic Microwave Background (CMB) radiation. You may assume that the CMB radiation is a black-body radiation with a temperature $T_{\mathrm{CMB}} \approx 2.8\,\mathrm{K}$ that fills the entire universe. The black hole will therefore lose energy from Hawking radiation and absorb energy from the CMB radiation according to the Stefan-Boltzmann law.

(xii) [1 point] When a black hole reaches a certain critical mass M_C, the net power radiated by the black hole is exactly zero. This is often called the equilibrium condition. Calculate this critical mass in kg. Name an astronomical object which has a similar mass, up to one order of magnitude.

(xiii) (a) [0.5 points] Is the equilibrium mentioned above in Part (xii) stable or unstable?

(b) [0.5 points] Comment on the value obtained in Part (xi) in relation to the above conclusion.

Question 3
The Red Distance
(Total: 20 points)

Introduction

It is a well-known fact that nothing travels faster than light (except for fictional objects like the Starship Enterprise). Consequently, in light of the age of the universe being approximately 13.7 billion years, it comes as a surprise to most first-time astronomers to learn that the radius of the observable universe is approximately 46.5 Gly.

Of course, astronomers like those in this competition know that this discrepancy is due to the expansion of space. In this question, we will look at how expansion of space affects our intuitive notion of distance.

Part 1: Relativistic Redshift (Sub-total: 7 points)

Distances of objects near the edge of the observable universe are often difficult to measure precisely, due to observational limitations. For this reason, amongst others, distances from Earth to such objects are often measured in terms of their *relativistic redshifts*. The relativistic redshift z of a moving object in the radial direction is given as

$$1 + z = \gamma \left(1 + \frac{v}{c}\right),$$

where γ is the Lorentz factor and v is the velocity of the object moving away from the observer in the radial direction.

(i) [2 points] There are three main types of redshifts, but the redshift we are primarily concerned with in this question is *cosmological redshift*. Explain the phenomena of cosmological redshift.

Using this formula, one can determine "distances" from the observer to the object. This is done by interpreting cosmological redshift as a receding velocity. That is to say, suppose we had an object at an extreme distance. To this distance we may associate a "rate at which the object is moving away from us" due to the expansion of space. This rate of recession is expressed as a receding velocity, and we may utilise the relativistic redshift formula to compute the proper distance from the object to us.

(ii) [3 points] Assume that a distant object is completely stationary, such that any perceived motion is solely due to cosmological expansion. Prove that the (proper) distance d from the observer to the object is given by

$$d = C \cdot \frac{(1+z)^2 - 1}{(1+z)^2 + 1},$$

where C is some expression in terms of known quantities to be determined. In your answer, you should state an expression for C, with reference to the Formula Booklet, or otherwise.

(iii) [2 points] Hence or otherwise, explain why C cannot remain constant with time. Compute the value of C today based on the formula booklet.

Part 2: Let's Hubble Along **(Sub-total: 4 points)**

The concept of measuring a "distance" takes a rather interesting turn, pun intended, when we approach the edge of the observable universe. To further define distances in a proper fashion, however, we first need to consider the Hubble parameter H. The Hubble parameter is a function of the redshift z, and so we write it $H(z)$ without loss of generality. The derivation of the Hubble parameter comes from Einstein's field equations, but we save you all this work and give you the final result. It is given by

$$(H(z))^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3},$$

where G is the gravitational constant, ρ is the mass density of the universe, k is the normalised spatial curvature of the universe, Λ is the cosmological constant, and $a = \frac{1}{1+z}$ is the scale factor. This is also known as Friedmann's second equation.

A bit needs to be said about the scale factor a. It relates the proper distance (see Part 3 for an explanation) between two objects. If at the present time we receive light from a distant object with redshift z, the scale factor at the time the light originated from the object is $a = \frac{1}{1+z}$.

(iv) [1 point] Explain why the relation $d(t) = a(t)d_0$ holds, where $d_0 = a(t_0)$ is the distance at some reference time, and t is time taken with reference to t_0.

(v) [3 points] Suppose that the universe is matter-dominated with density of matter today at ρ_0. It is given that

$$\rho_c = \frac{3H_0^2}{8\pi G}, \qquad \Omega_m = \frac{\rho_0}{\rho_c}, \qquad \Omega_k = -\frac{kc^2}{H_0^2}, \qquad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}.$$

Here, H_0 is the Hubble constant today. Show carefully that

$$H(z) = H_0 E(z),$$

where

$$E(z) = \sqrt{B(1+z)^3 + C(1+z)^2 + D},$$

and B, C, D are constants to be determined in terms of the four quantities provided above.

Part 3: Too Far Away! **(Sub-total: 9 points)**

There are three types of distances we shall now contend with: the *proper distance*, the *comoving distance*, and the *light travel distance*.

The proper distance between two objects is the separation of the two objects measured at a specific cosmological time. Loosely speaking, it is the distance between the objects factoring in cosmological expansion. The proper distance changes with time.

The comoving distance, on the other hand, is the separation of the two objects measured at the current cosmological time. Loosely speaking, it is the distance between the objects factoring out cosmological expansion. The comoving distance does not change with time. To define a comoving distance, we need to fix a time T and measure proper distance at that time T. That becomes the comoving distance.

In short, while proper distance can be likened to a movie, the comoving distance can be likened to taking a frame, or a snapshot, of an instant in the said movie.

The light travel distance is the time taken for light to reach from the object to us, multiplied by the speed of light.

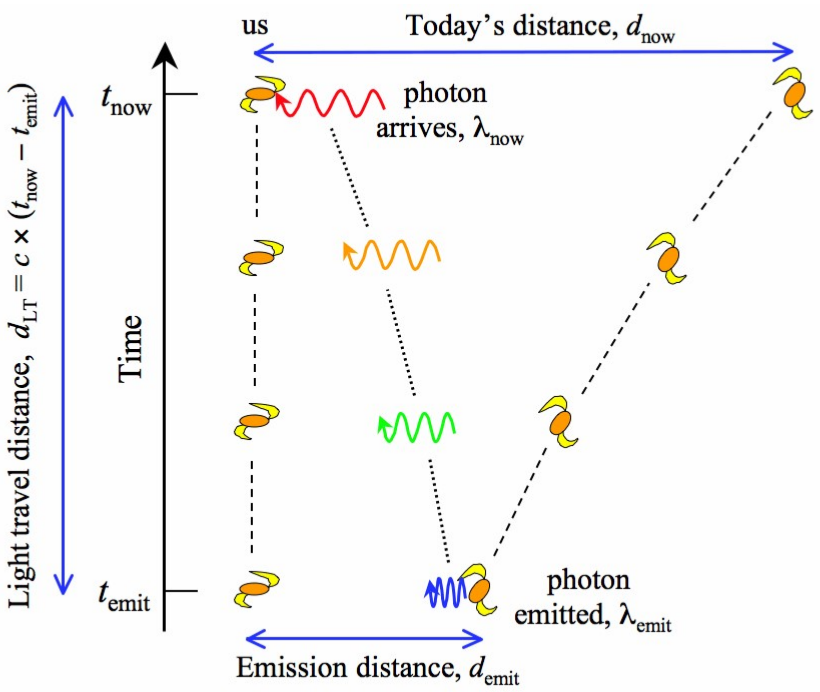

Figure 3: A nice illustration of the three distances we found on Google. Here, emission distance is the comoving distance with respect to t_{emit}, and also the proper distance at that time. Today's distance is the proper distance today.

(vi) [1 point] Using a simple example, or otherwise, demonstrate that light travel distance is distinct from the proper distance.

(vii) [4 points] Suppose the Hubble constant today is H_0. Let $d_H = \frac{c}{H_0}$ be the Hubble distance. Show that the comoving distance d_C satisfies

$$d_C(Z) = d_H \int_0^Z \frac{1}{E(z)}\,dz,$$

where $d_C(Z)$ is the comoving distance of an object with cosmological redshift Z.

(Hints: One way to do this is to start by considering expressing d_C using c and a. Note that for a given object, the redshift Z is a function of time, and vice-versa (since space is expanding). At some point in your answer, you may find the relation $H(t) = \frac{a'(t)}{a(t)}$ useful.)

(viii) [3 points] Find a similar expression for the light travel distance $d_T(Z)$.

(Hint: At some point in your answer, you may find the following rule useful. If $f(x) = \int_0^x g(s)\,ds$, then $f'(x) = g(x)$.)

(ix) [1 point] Explain why the age of the universe is given as

$$\lim_{z\to\infty} \frac{d_T(z)}{c}.$$

Question 4
A Study of the Big Dipper
(Total: 20 points)

Part 1: Introduction **(Sub-total: 8 points)**

The Big Dipper is well known across many cultures since antiquity. Some details about the Big Dipper are provided below.

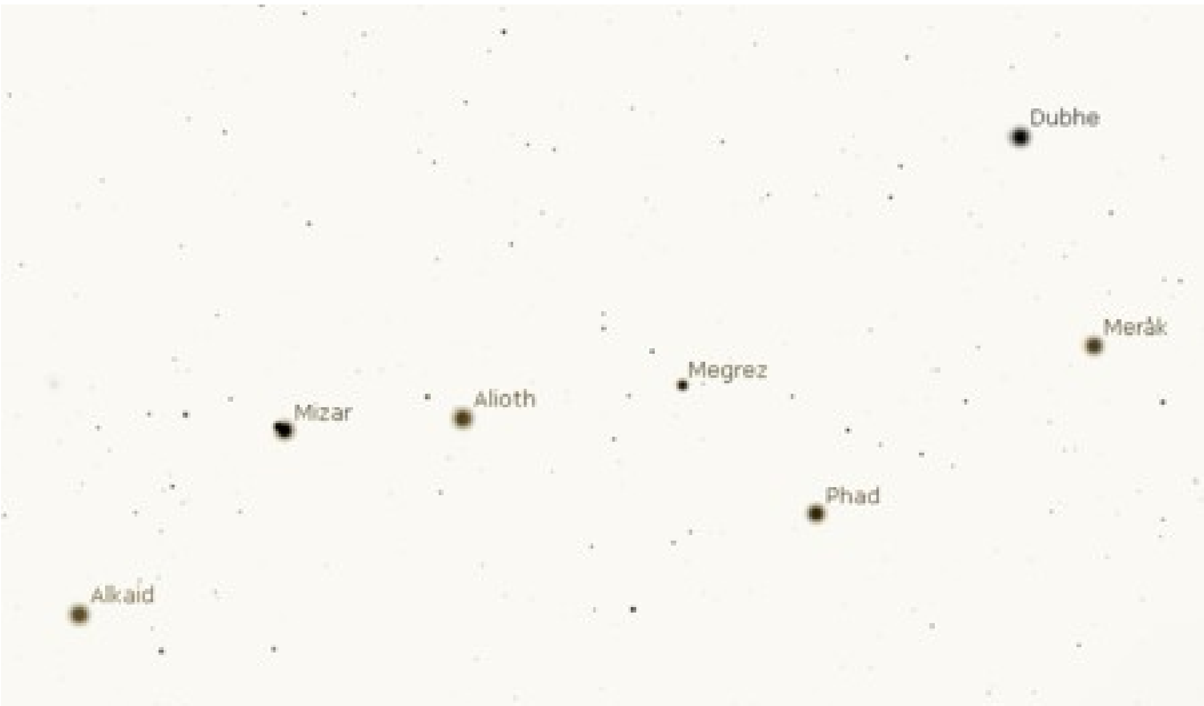

Figure 4: The Big Dipper.

Star	**Apparent Magnitude** m	**Right Ascension**	**Declination**
Dubhe	+1.79	11h 03min	$+61^\circ\,45'$
Merak	+2.37	11h 01min	$+56^\circ\,22'$
Phad	+2.44	11h 53min	$+53^\circ\,41'$
Megrez	+3.31	12h 15min	$+57^\circ\,01'$
Alioth	+1.77	12h 54min	$+55^\circ\,57'$
Mizar	+2.27	13h 23min	$+54^\circ\,55'$
Alkaid	+1.86	13h 47min	$+49^\circ\,18'$

Table 2: Stars of the Big Dipper.

(i) Modern astronomers label the Big Dipper as an *asterism.*

(a) [1 point] Define an asterism.

(b) [1 point] State an example of an asterism other than the Big Dipper.

(ii) [2 points] It is well known that one can find Polaris simply by extending a line from Merak to Dubhe. This is merely one of the ways where the Big Dipper serves as an important signpost to other bright stars. Explain how one can use the Big Dipper to find two other bright stars (except Polaris).

(iii) The Big Dipper is placed in the far northern sky, making it difficult for southern observers to see. Below is a list of four cities in Australia and New Zealand.

City	**Latitude**	**Longitude**
Hobart, AU	$42° 52' S$	$149° 19' E$
Brisbane, AU	$27° 28' S$	$153° 02' E$
Auckland, NZ	$36° 50' S$	$174° 44' E$
Wellington, NZ	$41° 17' S$	$174° 46' E$

Table 3: Cities in Australia and New Zealand.

(a) [1.5 points] Which of these cities cannot see any star of the Big Dipper?

(b) [1.5 points] Which of these cities can only see part of the Big Dipper over the course of an entire day?

In answering the above two parts (a) and (b), you should show your working. You may ignore atmospheric extinction and assume a flat horizon.

(iv) [1 point] To the naked eye, the stars of the Big Dipper have noticeably different brightnesses. How much brighter is the brightest star of the Big Dipper compared to its faintest star? Express your answer in percentages.

Part 2: Starry Night Over the Rhône (Sub-total: 12 points)

The Big Dipper is prominently featured in Vincent Van Gogh's *Starry Night Over the Rhône*. Painted in September 1888, it features the city of Arles, France at night.

Figure 5: Van Gogh's *Starry Night Over the Rhône*. Truly a work of art.

But what time is it? Using the night sky depicted here, we can recover the time of night that Van Gogh was trying to faithfully depict.

(v) [2 points] By matching the features of the cityscape with actual landmarks, we know that this was painted from Arles at the coordinates $43^\circ\,41'\,$N, $4^\circ\,38'\,$E. From this point, the Big Dipper is circumpolar. What is the lowest altitude reached by any part of the Big Dipper when it skims the horizon?

(vi) For simplicity, let us assume that this work was painted on the autumnal equinox (September 22).

  (a) [1 point] On this date, what is the approximate right ascension of the Sun to the nearest minute?

  (b) [1 point] Briefly explain your answer.

(vii) [2 points] Define the hour angle as the number of hours since the object passed the upper local meridian. This means that when an object is at the highest point in the sky, it has an hour angle of 0h 00min.

With this in mind, estimate the hour angle of Merak/Dubhe to the nearest minute. In your answer, you should state a suitable assumption/simplification that you need to make.

For the rest of this DRQ, you may treat Merak and Dubhe as sharing the same right ascension of 11h 02min.

(viii) [1 point] Hence or otherwise, determine the hour angle of the Sun during the moment depicted in this painting.

(ix) [2 points] For simplicity, let us define sunset as the point in time when the apparent centre of the Sun touches the horizon, ignoring atmospheric refraction. In Greenwich, UK, sunset on September 22nd occurs at 1753h (Greenwich Mean Time, a.k.a. GMT). When would sunset be if we were displaced $4^\circ\, 38'$ E from Greenwich? Give your answer in GMT.

(x) [2 points] An astute observer notes that Arles and Greenwich are at different latitudes. Hence, the answer found in Part (ix) may not necessarily correspond to the actual sunset time at Arles in GMT. Is this a major concern? Explain your answer.

(xi) [1 point] Assume your answer in Part (ix) is correct. Hence, find the exact time depicted in this painting, in GMT.

Question 5
Alice and Bob Circling in the Sky
∼A Story of Binary Star Systems∼
(Total: 20 points)

Introduction

In a faraway land in outer space, in the constellation of Gemini lies the two stars, Alice and Bob. This is a love story of Alice and Bob, two binary stars circling in the heavens above. From our astronomical observations, it is found that most stars indeed do come in pairs (as most good things do), unlike our lonely parent star, the Sun. The study of these binary systems is hence an important research in understanding the formation of star systems in nebulae.

In the following questions, you may assume that the plane of orbit is parallel to the plane of observation.

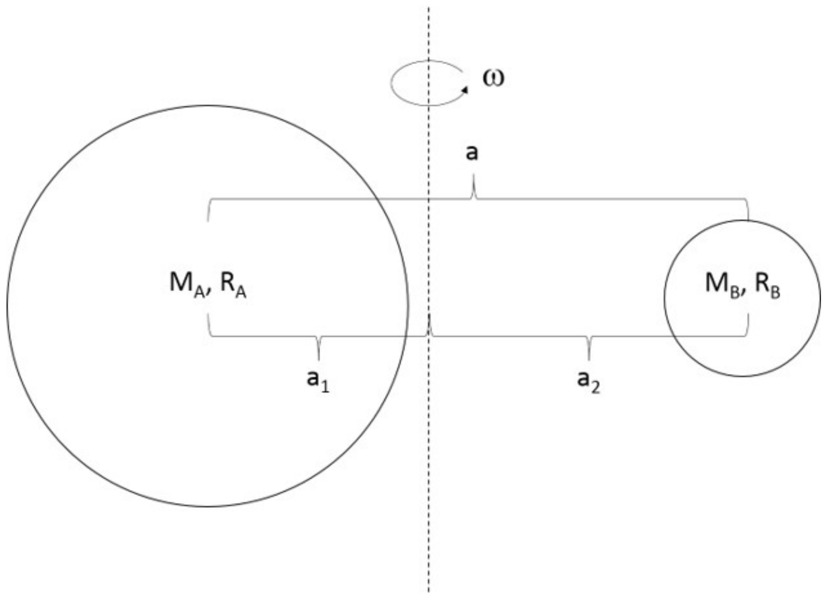

Figure 6: Alice and Bob (not drawn to scale) circling.

Part 1: Celestial Mechanics (Sub-total: 9 points)

Let Alice and Bob have mass M_A and M_B, and radius R_A and R_B respectively, as shown in the diagram (the subscripts A and B refers to Alice and Bob respectively). Note that Alice and Bob have different masses $M_A > M_B$.

(i) [2 points] The distance to the centre of mass as measured from the primary star Alice is a_1. Express a_1 in terms of the distance a between the two stellar centres, M_A, and M_B.

Consider the following graph of radial velocity against phase of Alice and Bob.

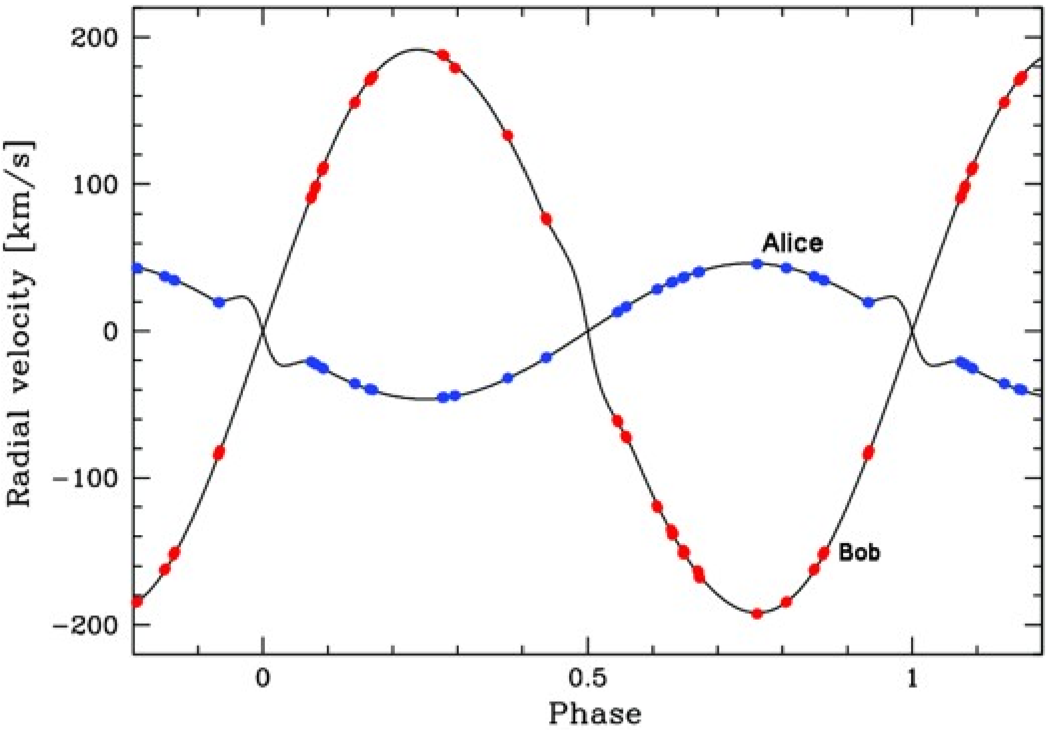

Figure 7: Alice and Bob's phase graph.

It is found from observation that the distance between the two stars (centre of mass to centre of mass) is 0.062 AU.

(ii) [3 points] Determine the masses M_A and M_B of Alice and Bob respectively in solar units.

(iii) [3 points] It was found that the surface temperature of Alice and Bob are 13000 K and 4500 K respectively. Using the results in Part (ii) and the Stefan-Boltzmann law, determine the radius R_A and R_B of Alice and Bob in solar units. State all assumptions used in your calculations.

(iv) [1 point] Why can we use the Stefan-Boltzmann law in our calculation?

Part 2: Gravitational Waves **(Sub-total: 11 points)**

In another universe (if you buy into the idea of the multi-verse), Alice and Bob are in fact two neutron stars that are spiralling into each other. The story of Alice and Bob could develop into either of two fates: they merge to form a new neutron star, or they merge and become a black hole. Understanding the fate of the merger requires us to develop an understanding of the in-falling process.

Data of Alice and Bob was collected from ground base observatories, monitoring their dynamical relationship. It is understood that due to the densities of the neutron stars, the orbit has an initial period P_0 of approximately 7.752 hours. Furthermore, Alice is in fact a pulsar, emitting strong electromagnetic radiation in the radio-wave spectrum. We observe instances that Alice returns to the periastron, and call that the *Time of Periastron Passage*. Given that the orbit is decaying, the period P_n after n periastron passages is smaller than the initial value P_0.

(v) [2 points] The periastron refers to the point of closest approach between the binary stars. It is equivalent to the perigee of a satellite (natural or man-made) orbiting the planet. Using this information, sketch a diagram depicting the orbit of the two neutron stars. In your diagram, you should label the periastron and apastron.

(vi) [1 point] Suggest a possible cause of the orbital decay.

From theoretical calculations, the time between the 0^{th} and N^{th} periastron, T_N, can be expressed as an equation

$$\frac{T_N}{P_0} = \sum_{i=1}^{N} (1 + \dot{P})^i.$$

Note that $\dot{P}$ is the time derivative of P, the orbital period.

(vii) [2 points] Expand the summation and show that given that $\dot{P}$ is extremely small, the summation may be approximated as

$$\frac{T_N}{P_0} = N + \frac{N(N+1)}{2}\dot{P}.$$

(viii) [6 points] Hence, the change in the orbital period after i periastrons is

$$\Delta t_i = T_i - T_0 = \frac{i(i+1)}{2} P_0 \dot{P}.$$

Complete Table 4, linearise the above equation, and plot an appropriate graph to obtain the observed value of P.

(*Note: You should detach the table provided and attach it to your answer script.*)

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Detach this page and attach it to your answer script.

Approximate Date	**Time of Periastron Passage (JED – 2,440,000)**	**Number of Hours Between 2 Observations**	**Number of Complete Orbits Between 2 Observations**	**Actual Cumulative Completed Orbits**	Δt_i
1974.77	2331.446	0	0	0	0
1974.93	2389.586	1395.349	180	180	−0.000883681
1976.13	2826.924	10496.126	1354.000036		−0.063867225
1976.93	3118.591	7000.001	903.0000096		−0.161151198
1977.58	3356.640	5713.179	737.0000111		−0.27333482
1977.96	3493.591	3286.822	424.0000088		−0.351226531
1978.23	3593.397	2395.349	309.0000019		−0.414135323
1978.42	3663.488	1682.171	217.0000065		−0.46140991
1978.82	3807.544	3457.365	446.0000028		−0.566593685
1979.31	3988.423	4341.086	560.0000059		−0.713943244
1980.10	4276.537	6914.730	892.0000106		−0.983779849
1980.59	4454.509	4271.319	551.0000043		−1.172027064
1980.59	4455.477	23.256	3.000000635		−1.173097083
1981.14	4656.382	4821.706	622.000003		−1.405491826

Table 4: Table for Part (viii).

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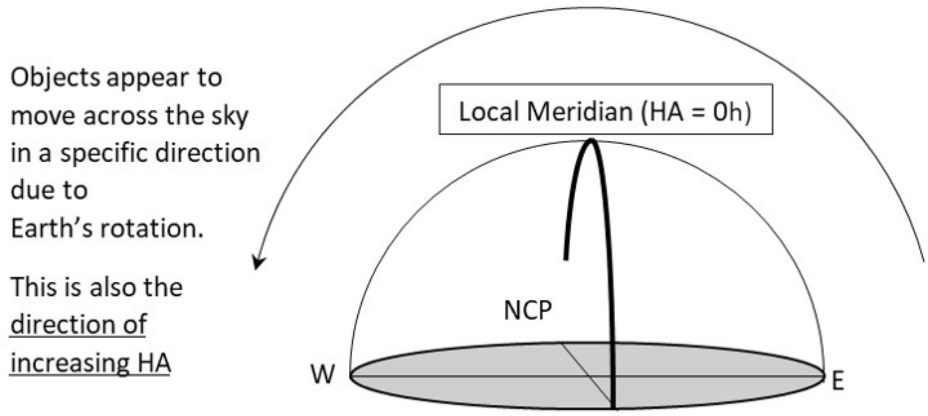
Appendix A

(Adapted from AC 2015 SNR DRQ Q5: Seasons in the Sun)

In astronomy, the concept of the **Hour Angle (HA)** is often used. In general, the HA is the angular distance of a point measured westward from the Local Meridian (the upper arc connecting the zenith to the celestial poles).

The HA can be expressed in terms of angles (degrees/radians/arcseconds etc.), or units of time like hours/minutes/seconds. If we adopt the latter convention, the HA represents the amount of time since the object last crossed the meridian. Indeed, hour angles are often expressed in this form. A diagram for an observer in the Northern Hemisphere is shown below. For clarity, the South Celestial Pole has been omitted.



Determining the HA for any object at a specific time can be computationally demanding. However, calculations for the Sun are greatly simplified if we use solar time, as we can use more natural units. For example, the Sun has an HA of 0 hours at solar noon (as it must be on the Local Meridian then).