AstroChallenge 2018
Data Response Questions
(Junior)

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. This paper consists of **22** printed pages and **6** blank pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your school (and team number) on **EVERY** page, as well as the individuals in the said team on the first page.
6. It is your team's responsibility to ensure that all pages of your answer script have been submitted, **including pages to be detached from this booklet**.

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Question 1
Do You See What I See?
(Total: 20 points)

Part 1: The Many Eyes of Astronomy (Sub-total: 11 points)

The layman is familiar with using binoculars and telescopes in the backyard to look at stars. This is an example of *optical astronomy*, i.e. astronomy in the visible light spectrum. We also call this *visible light astronomy*. Astronomy, however, comes in many different forms, across the entire electromagnetic spectrum, and more.

The electromagnetic spectrum describes the range of electromagnetic waves in terms of their wavelength. Broadly speaking, there are seven distinct classes of electromagnetic waves.

Class	Average wavelength
Gamma	1 pm
X-ray	500 pm
Ultraviolet	100 nm
Visible	500 nm
Infrared	10 μm
Microwave	1 mm to 10 cm
Radio	10 m

Of course, these classes are really "bands", so each class really establishes a range on the electromagnetic spectrum.

Astronomy takes place across all these bands. Many objects in the universe (e.g. gas clouds, black holes) are not visible to the naked eye. In other words, they emit very little, if at all, radiation in the visible light spectrum. However, any object with heat does emit radiation. Therefore, to obtain a good picture of the universe, we need to consider the entire electromagnetic spectrum. In fact, most images one might see of beautiful objects in the universe are false colour images. If one were to look at these objects in true colour, one should expect to literally see space, because our eyes cannot detect the radiation these objects emit.

(i) [2 points] Provide, with clear justification, a use for microwave astronomy and a use for X-ray astronomy.

Solution. We will use Wien's displacement law to justify our answers. We have

$$T = \frac{2.9 \times 10^{-3}}{\lambda},$$

and thus

$$T_m = \frac{2.9 \times 10^{-3}}{1 \times 10^{-3}} = 2.9\,\mathrm{K}, \qquad T_X = \frac{2.9 \times 10^{-3}}{500 \times 10^{-12}} = 5.8 \times 10^6\,\mathrm{K}\,.$$

Thus microwave astronomy could be used for analysis of CMB radiation, and X-ray astronomy could be used for analysis of... any answer with really high temperatures of that approximate order will do. Supernovae, intermediate mass X-ray binaries, black hole accretion disks, etc.

(Note: While the "justification" used is Wien's displacement law, it was not necessary to perform this computation. Answers demonstrating understanding of use were awarded full credit.)

Due to the varying wavelengths of each band, different types of telescopes are needed to perform different types of astronomy. For example, huge satellite dishes are used to perform radio astronomy, standard optical telescopes are used to perform optical astronomy, and gamma ray detectors used to perform gamma ray astronomy.

(ii) [1 point] Explain why radio telescopes need extremely large receivers (satellite dishes).

Solution. There are two main reasons.

The first main reason is because radio sources are typically weak.

The second main reason (and more significant, probably) is because of the large wavelength of radio waves. The angular resolution of a telescope is proportional to the wavelength of light collected divided by the diameter of the collection surface. To achieve a good angular resolution, the receiving dish needs to be large since the wavelengths of radio waves are extremely large.

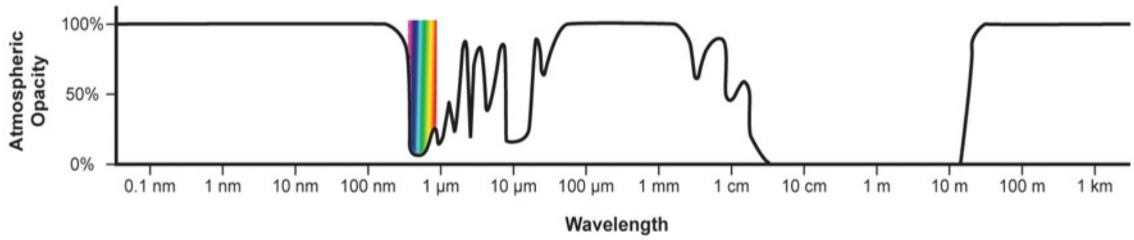
(iii) [2 points] State and explain two difficulties of gamma ray astronomy if we were to try to use the conventional method of focusing light to detect gamma rays.

Solution. Difficulty 1: Long time needed to perform a detection. Gamma rays are significantly rare even when considering bright sources, so it takes time for a small and limited collection radius to collect enough gamma rays to make a detection.

Difficulty 2: Difficulty in focusing gamma rays. Gamma rays penetrate deeply into most materials and are not reflected. Gamma rays also have wavelength about 1 picometre, which causes diffraction and scattering effects to be very significant when it interacts with matter (since atomic radius is assumed at 1 angstrom). Conventional refraction and reflection don't work in this case.

Note that there are crystals which can bend light using Bragg diffraction. But it is very hard to focus gamma rays to a point merely using this method.

One additional difficulty with performing astronomy in each of these ranges is due to *atmospheric opacity.* Gases in Earth's atmosphere prevent certain wavelengths of electromagnetic radiation from permeating the atmosphere. Because of this, certain telescopes and detectors cannot be built on the ground, and must instead be put in space.



(iv) [3 points] The atmosphere is completely opaque to gamma radiation. Yet there are ground-based gamma ray detectors. How do they work, and why are they in fact superior to space-based gamma ray telescopes?

Solution. Gamma rays interact with molecules in the atmosphere to produce a number of secondary particles. These particles are what is sought-after by ground-based detectors. Two main methods of detection exist – by radiation counters, and by observing the Cherenkov radiation (since these secondary particles are also relativistic).

They are superior to space-based gamma ray telescopes because of the larger detection range. Using ground-based detectors, the sky is (literally) the limit, because the atmosphere itself acts as the "collector", in contrast to the small collection radius of space-based gamma ray telescopes.

(v) [1 point] What is an advantage to using space-based telescopes as compared to ground-based telescopes for optical astronomy? Explain.

Solution. Space-based telescopes are not subject to atmospheric blurring. Atmospheric blurring refers to the blurring of atmospheric objects due to turbulence in Earth's atmosphere. In space, telescopes are subjected to no such effects.

Finally, we may of course study the same object in space using different ranges of the electromagnetic spectrum to gain for information about the object. Additionally, even when we expect objects to emit radiation in some range of the electromagnetic spectrum, this may not be the case. Such is the case with *embedded star clusters.* A typical example of an embedded star cluster is shown below.

Figure 1: Orion Trapezium Cluster, in optical (left) view and in infrared (right) view.

(vi) [1 point] What are embedded star clusters?

Solution. Groups of very young stars partially or completely enclosed in interstellar clouds of dust and/or gas.

(vii) [1 point] Explain why infrared astronomy has been very useful in detecting such star clusters.

Solution. Since the stars are enclosed, we sometimes cannot detect them using optical methods. Infrared has much higher penetration through dust and so for some of these star clusters, we can detect the stars emitting infrared.

Part 2: Another Spectrum (Sub-total: 9 points)

Basics of Spectra

Another very important tool used when studying astronomical objects is the *spectrum* of the object. As an example, here is a sample absorption spectrum of a star HD 145482.

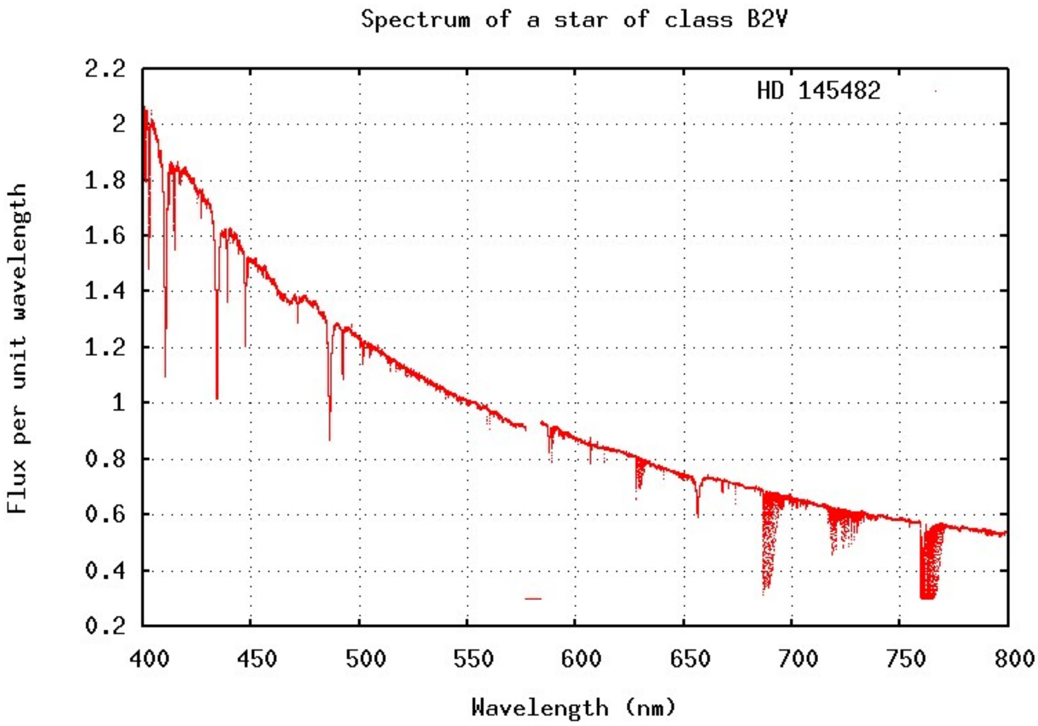
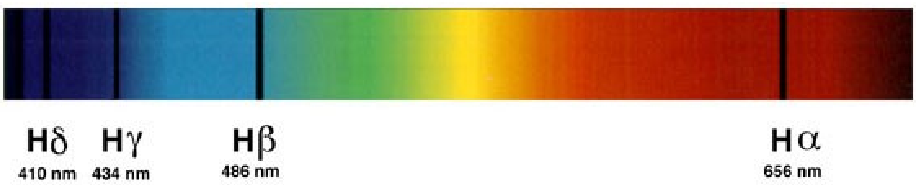
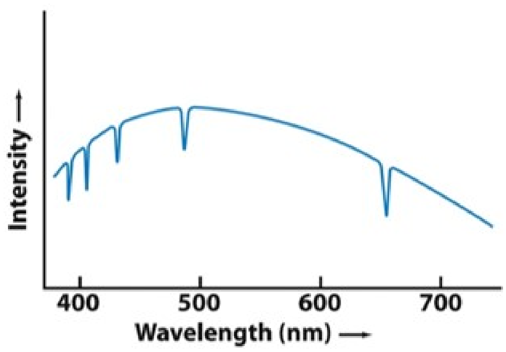


Figure 2: Absorption spectrum of HD 145482.

Each of these dips is called an *absorption line*. These absorption lines yield a wealth of information about the star. Each element and/or molecule is associated with a set of wavelengths. For example, the absorption lines of hydrogen (and the associated graph) are as follows.



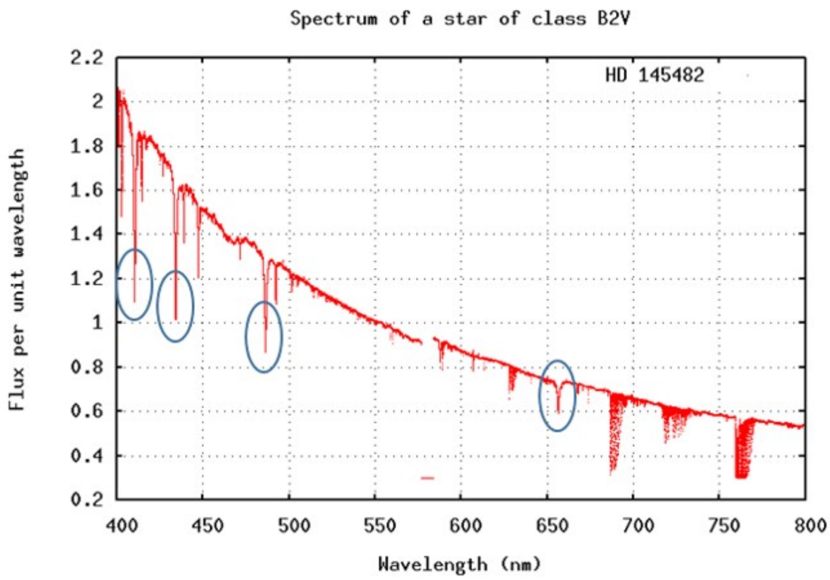


By studying the absorption lines of a stars spectrum, one can obtain information about the composition of the star by checking for the characteristic lines of the spectra of the element or molecule in question. We can furthermore deduce properties based on the elements and molecules present, for example ongoing fusion processes. Rotational properties can be deduced from the broadening of the spectral lines, although these can be caused by collisions (known as collision damping) and thermal movement as well. Horizontal scaling of spectral lines suggests radial movement of the star.

(viii) [1 point] A copy of the absorption spectrum graph of HD 145482 is attached to the back of this question. Identify the hydrogen lines $H\alpha, H\beta, H\gamma, H\delta$ in the spectrum of the star HD 145482, and indicate them on this graph.

(*Note: You should detach the graph provided and attach it to your answer script.*)

Solution.



(ix) [2 points] There are in fact two types of spectra – absorption spectra and emission spectra. Since absorption spectra involves light passing through a gas, and an emission spectra looks at diffracting emitted light, one might expect us to use emission spectra to study stars instead of absorption spectra. Why do we use absorption spectra instead of emission spectra?

Solution. We use absorption spectra because the plasma (which is really ionised gas) of the star already absorbs light emitted from the star. This gives us information about the compounds present in the star.

In contrast, we in fact cannot use an emission spectrum, for the simple reason that the star outputs a range of frequencies of light by virtue of its temperature, instead of light only of certain frequencies caused by de-excitation of its molecules. If we diffracted this light we would simply get an absorption spectrum.

(x) [1 point] HD 145482 is in fact a spectroscopic binary. What is a spectroscopic binary, and how would you determine that HD 145482 is a spectroscopic binary?

Solution. A binary system whose components are inferred by the redshifting and blueshifting of their spectral lines.

The spectral lines would shift over time periodically, due to the Doppler effect as the stars orbit each other.

Note: Reject anything to do with high zoom resolving the stars, tracking their motion this way, etc.

Doppler Line Broadening

Doppler broadening due to thermal effects may be modelled by the Maxwell-Boltzmann distribution. Informally speaking, the random motions of a gas depend on temperature, and the distribution of movement of molecules thus also depends on temperature. By the Doppler effect on molecules in a gas, therefore there will be a range of wavelengths (and hence frequencies) detected by spectroscopic methods. The profile function for the distribution of frequencies due to thermal effects is given by

$$\phi(f) = \frac{1}{\Delta f_D \sqrt{\pi}} e^{-\frac{(f-f_0)^2}{(\Delta f_D)^2}},$$

where $\Delta f_D = \frac{f_0}{c}\sqrt{\frac{2kT}{m}}$ is the *Doppler width*. Here, f_0 is the average (mean) frequency, c is the speed of light, k is the Boltzmann constant, T is the temperature of the gas, m is the particle mass, and $e = 2.71828\ldots$ is Euler's number, a constant.

With the formula $\phi(f)$ above, the *full width at half maximum* (FWHM) is the width of $\phi(f)$ at half its maximum value. In layman's terms, the FWHM describes the average spread of the function. Graphically, the FWHM is seen as follows.

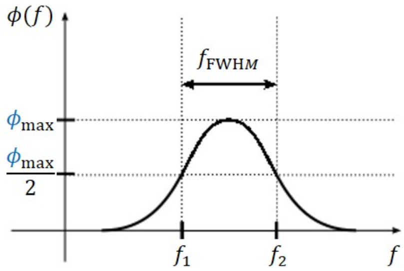


Figure 3: FWHM of $\phi(f)$.

Where there is a spectral line at f_0, the function $\phi(f)$ will describe the broadening of the line. The FWHM translates to the width of the line, i.e. at half-maximum the line will range from $f_1 = f_0 - \frac{f_{\text{FWHM}}}{2}$ to $f_2 = f_0 + \frac{f_{\text{FWHM}}}{2}$, where f_{FWHM} is the FWHM of $\phi(f)$.

The Doppler width is extremely important, as it determines the influence of the considered effect (in this case, thermal) on the width of the broadened spectral line. In the questions to follow, you will see how this is true.

(xi) [2 points] The maximum of $\phi(f)$ is $\phi(f_0)$. Find the FWHM of $\phi(f)$. Express your answer in terms of Δf_D.

Solution. The maximum is $\frac{1}{\Delta f_D \sqrt{\pi}}$. Hence we are finding f such that

$$\frac{1}{2\Delta f_D \sqrt{\pi}} = \frac{1}{\Delta f_D \sqrt{\pi}} e^{-\frac{(f-f_0)^2}{(\Delta f_D)^2}}.$$

Simplifying,

$$\frac{1}{2} = e^{-\frac{(f-f_0)^2}{(\Delta f_D)^2}}.$$

Taking logarithms, we have $\ln 2 = \frac{(f-f_0)^2}{(\Delta f_D)^2}$. Thus,

$$f = f_0 \pm \Delta f_D \sqrt{\ln 2}.$$

Hence the FWHM is $2\Delta f_D \sqrt{\ln 2}$.

(xii) [1 point] Determine Δf_D in terms of wavelength λ.

Solution. Since $c = f\lambda$ for electromagnetic waves, we have

$$\frac{\frac{c}{\lambda_0}}{c}\sqrt{\frac{2kT}{m}} = \frac{1}{\lambda_0}\sqrt{\frac{2kT}{m}}.$$

(xiii) [2 points] Assume the Sun has temperature 6000 K. The effect of collision damping on the width of the Balmer lines of hydrogen is approximately 0.1 pm, i.e. the Balmer lines are broadened about the mean by approximately 0.1 pm due to collision damping.

Hence, or otherwise, prove that the broadening of the Balmer lines due to thermal effects dominates the broadening due to collision damping. You may assume the mass of hydrogen is $1.0078u$. Here, u is the atomic mass unit.

Solution. For hydrogen, we have

$$\begin{aligned}\frac{1}{\lambda_{\text{FWHM}}} = \frac{2\sqrt{\ln 2}}{400 \times 10^{-9}}\sqrt{\frac{2kT}{m}} &= \frac{2\sqrt{\ln 2}}{400 \times 10^{-9}}\sqrt{\frac{2 \times 1.3806488 \times 10^{-23} \times 6000}{1.0078 \times 1.660539 \times 10^{-27}}} \\ &= 2.52557 \times 10^{10}.\end{aligned}$$

Hence,

$$\lambda_{\text{FWHM}} = 2.4143 \times 10^{-11},$$

which is approximately 0.0241 nm. Clearly this dominates the collision damping effect.

(Note: The function ϕ has the property that the area under the curve from $\phi(f_1)$ to $\phi(f_2)$ is the probability that a random particle selected in the gas will have a Doppler-shifted frequency between f_1 and f_2. More advanced students will recognise ϕ as a probability distribution function known as a *Gaussian distribution*, although this fact is not needed in this question.)

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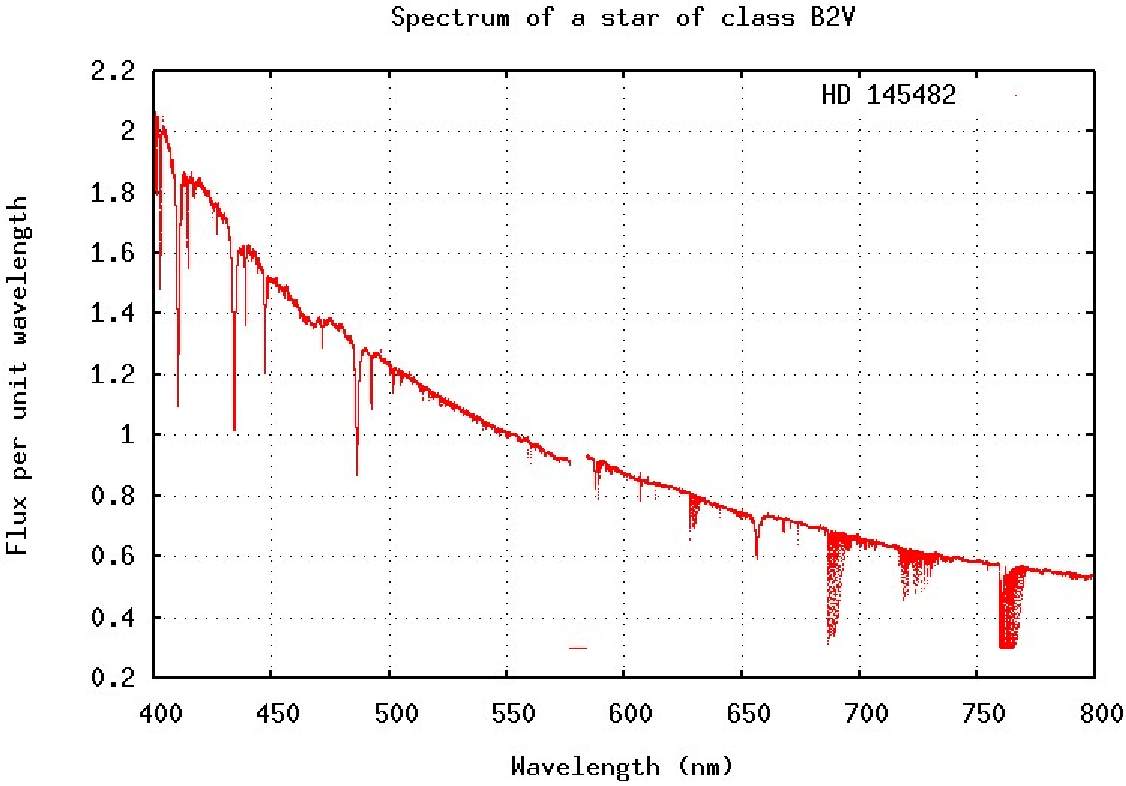

Figure 4: Copy of absorption spectrum of HD 145482.

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Question 2
Planetary Analysis
(21 points)

Introduction

On 14 December 2017, NASA and Google announced the discovery of an eighth planet, Kepler-90i, in the Kepler-90 system. The discovery was made using a new machine learning method developed by Google. However, how much do we truly know about the exoplanet Kepler-90i, or in fact about any of the exoplanets that are currently being discovered in general? Let us find out more by investigating the underlying geophysical concepts and the astronomy concepts revolving around the structure of a planet itself, as well as its orbital characteristics. Furthermore, the chances of life on Kepler-90i will be discussed at the end of this question.

Part 1: Radiogenic Heat and Conceptual Preliminaries (Sub-total: 4 points)

Kepler-90 system is a star system formed 2 billion years ago. Hence, we would thus expect the primary source of internal heat to be from radioactive decay. For this part of the question, we will investigate the physics behind radioactive decay and how it becomes the primary heat source a few billion years after the formation of the star system.

(i) [1 point] We tend to assume that one primary heat source for a planet a few billion years after the formation of the star system is via radioactive decay. Briefly explain why this assumption is valid.

Solution. This is because heat deposited via other means (primordial heat) are temporary and only for a short period of time (i.e via gravitational contraction, accretion, differentiation etc.), while radioactive decay continues to provide heat over time (radiogenic heat). Hence, over a long period of time, usually over billions of years, the total amount of heat deposited by radioactive decay will be comparable to or greater than the amount that was deposited initially.

In any radioactive decay reaction, they tend to follow the universal law of radioactive decay. This law, in summary, states that the decay of an unstable nucleus is entirely random, and it is impossible to predict when a particular atom will decay. Hence, we can expect to model this as a rate equation

$$\frac{\Delta N}{\Delta t} = -\lambda N,$$

where N refers to the number of unstable nuclides, λ is a positive constant (also known as the decay constant), and $\frac{\Delta N}{\Delta t}$ is the rate at which the number of unstable nuclides N changes. Since $\lambda > 0$, we would expect that N decreases with time.

The solution to this rate equation is given by

$$N = N_0 e^{-\lambda t},$$

where N_0 denotes the initial number (at $t = 0$) of unstable nuclides.

The half-life of an unstable nuclide X (e.g. the half-life of uranium-235, or plutonium-238, etc.) is defined as follows. Given a sample of X, the half-life $t_{\frac{1}{2}}$ of X is the time taken for the number of nuclides of X in the sample to decrease to half the original amount.

(ii) [1 point] In a sample of nuclide X, there are N_0 initial nuclides of X. Suppose X has decay constant λ. Prove that $t_{\frac{1}{2}} = \frac{\ln 2}{\lambda}$.

Solution. From $N = N_0 e^{-\lambda t}$, substitute $t = t_{\frac{1}{2}}$ and $N = 0.5 N_0$. Simplify to get the required expression.

There are a few types of radioactive decay. Two of them are given below.

- α decay refers to the emission of a helium-4 nuclide (we call the helium-4 nuclide an *alpha particle*) and a daughter nuclide (the main product nuclide of the decay).
- β^- decay refers to the emission of an electron ${}_{-1}^{0}\beta$ and a daughter nuclide.

For instance, a uranium-238 nucleus decays via α decay to form a thorium-234 nucleus and an α particle by the following decay chain:

$$ {}^{238}_{92}\text{U} \rightarrow {}^{234}_{90}\text{Th} + {}^{4}_{2}\text{He}. $$

The energy released from this nuclear reaction is calculated using the Einstein's mass-energy equivalence equation

$$ E = (\Delta m)c^2, $$

where c is the speed of light and Δm is the net decrease in the total mass of the nuclei, calculated by taking the difference between total mass of the reactants and the products.

A *radioactive chain decay* is a process whereby the daughter nuclide is also unstable and thus decays further. This process usually generates an additional amount of heat. Since the half-life of the unstable daughter nuclide of this reaction is extremely fast relative to the half-life of the parent nuclide, it can be assumed that each radioactive decay of the parent nuclide releases energy from the radioactive decay of the nuclide itself plus the radioactive decay of the daughter nuclide(s).

Refer to Appendix A for the radioactive decay chain of uranium-238. The nuclide ${}^{206}_{82}\text{Pb}$ is stable in this chain reaction. The relevant data required for the remaining part of this section is also given in the following table.

Nuclide	**Mass (in u, the atomic mass unit)**
Helium-4	4.002602
Uranium-238	238.051
Thorium-234	234.044
Protactinium-234m	234.043
Uranium-234	234.041
Lead-206	205.974

Note that the mass of an electron can be found in the Formula Booklet.

(iii) [2 points] Determine the total energy released from the entire chain of radioactive decay of ONE uranium-238 nuclide, and show that it is approximately 7.9×10^{-12} J.

Solution. This is done by calculating the net mass defect and using the Einstein's mass-energy equivalence equation.

There are a total number of 8 α particles and 6 β^- particles emitted. Since the mass of uranium-238 and lead-206 is known, the mass defect is

$$\Delta m = 205.974u + 8 \times 4.002602u + 6 \times 9.10938 \times 10^{-31}\,\text{kg} - 238.051u = -0.05288u.$$

By Einstein's mass-energy equivalence equation,

$$E = (\Delta m)c^2 = (0.05288u)(2.9979 \times 10^8\,\text{m}\,\text{s}^{-1})^2 = 7.89 \times 10^{-12}\,\text{J}\,.$$

Note that in actual fact, the emission of an electron is usually accompanied by the emission of an electron neutrino, but this is not so important in the context of the question.

Part 2: Planetary Thermodynamics (Sub-total: 7 points)

To analyse the thermodynamics of a planet, we will try to illustrate the concepts by using Earth as an example. Beneath the surface of the Earth, radiogenic heat is being generated mainly by the radioactive decay of unstable nuclides such as ${}^{40}_{20}\text{K}$, ${}^{238}_{92}\text{U}$, and ${}^{232}_{90}\text{Th}$. The power generated by ${}^{232}_{90}\text{Th}$ is approximately equal to that of ${}^{238}_{92}\text{U}$, and is twice of that of ${}^{40}_{20}\text{K}$ today. Geoneutrino detectors can detect the decay of ${}^{238}_{92}\text{U}$ and ${}^{232}_{90}\text{Th}$, but are unable to detect the decay of ${}^{40}_{20}\text{K}$.

Hence, the power generated by the decay of ${}^{40}_{20}\text{K}$ is being detected by other similar means. It has been determined that the current total heat flux from Earth to space is 44.2 TW. Note that the prefix T here is Tera and represents 10^{12}.

The half-life of uranium-238 is 4.468 billion years. Radiometric dating has determined that uranium made up $3.4 \times 10^{-6}\%$ of the mass of the Earth when it was first fully formed approximately 4.5 billion years ago.

(iv) [3 points] Show that the power generated by the radioactive decay of ${}^{238}_{92}\text{U}$ is approximately 10 TW.

(Hint: Don't be scared! You can split this question into solving multiple mini questions.

- Uranium made up $3.4 \times 10^{-6}\%$ of the mass of the Earth when it was first fully formed approximately 4.5 billion years ago. What is the mass of uranium now? Consider Part 1 of this question.
- With the above, how many uranium nuclei, N_0, are there now?
- How can you then proceed to calculate the power generated from radioactive decay? (Hint for a Hint: State how P, $\frac{\Delta N}{\Delta t}$, and E for the decay of a nuclide all relate to each another.)

Points will be awarded for solving the mini-questions!)

Solution. First, take note that the power generated from radioactive decay is given by

$$P = \frac{\Delta N}{\Delta t} \times E,$$

where E is the energy generated per uranium-238 nuclide decayed.

Recall that by the universal law of radioactive decay, $\frac{\Delta N}{\Delta t} = \lambda N$, where N is the number of radioactive nuclides.

Since approximately 1 half-life of uranium has occurred, we then know that $N = 0.5N_0$. The mass of one uranium-238 nuclide is given by $m_U = 238.051 \times 1.6605 \times 10^{-27}$ kg. The mass of the Earth is given from the formula booklet as $m_E = 5.972 \times 10^{24}$ kg.

Let R be the fraction of the mass of the Earth that is uranium-238. We see that

$$N_0 = \frac{m_E}{m_U} R.$$

Hence, by the results in Part 1,

$$P = 0.5\lambda N_0 E = \frac{0.5 R m_E E \ln 2}{t_{\frac{1}{2}} m_U}.$$

Solving for P, we get that P is approximately 10 TW.

(v) [1 point] Hence, show that the total power generated by the radioactive decay of ${}^{40}_{20}K$, ${}^{238}_{92}U$, and ${}^{232}_{90}Th$ is approximately 25 TW.

Solution. This previous part is the power generated by radioactive uranium-238. Hence, the total power generated by uranium-238, thorium-232 and potassium-40 is approximately $2.5 \times 10 = 25$ TW.

This is more than half of the total heat flux from Earth to space.

(vi) [1 point] Hence, comment on the assumption made in Part (i).

Solution. The total power generated by radioactive decay is approximately more than half of the total heat flux from Earth to space. This means that the assumption in Part (i) is valid, i.e. that the radiogenic heat as of today is one of the primary sources of internal heat inside the Earth.

(vii) [1 point] Explain briefly what equation below means to the layman.

$$P_{\text{Earth}} = \frac{(1-\varepsilon)(\pi R_E^2)}{4\pi d_{S-E}^2} L_S,$$

where

- P_{Earth} is the rate at which Earth receives energy from the Sun,
- ε represents the albedo of Earth,
- R_E represents the radius of Earth,
- d_{S-E} represents the Sun-Earth distance, and
- L_S represents luminosity of the Sun.

Solution. The amount of solar power absorbed is given by the fraction

$$\frac{\text{projected area of Earth} \times \text{albedo of Earth}}{\text{surface area od a sphere with radius the Sun-Earth distance}}.$$

(viii) [1 point] Determine, with suitable calculations, which source of heat is more important in determining the surface temperature of a planet that is relatively not very far from its own star, such as Earth itself. Earth has an albedo of approximately 0.30.

Solution. By substituting the relevant data from the formula booklet, $P_{\text{Earth}} = 1.2 \times 10^{17}\,\text{W} = 121800\,\text{TW} \gg 44\,\text{TW}$.

Hence, the main determinant of the surface temperature of the planet is the solar irradiance onto the planet's surface.

Part 3: Kepler-90i Orbital Characteristics and Chance of Life

(Sub-total: 10 points)

The following table describes some characteristics of the Kepler-90 star system.

Characteristic of Kepler-90	**Value**
Spectral type	G0V
Distance from Earth	2545 ly
Apparent magnitude V	+14
Absolute magnitude (M_V)	+4.3
Mass	$1.2M_\odot$
Surface temperature	6080 K
Age	Approx. 2 billion years

The following table describes some characteristics of the planet Kepler-90i, discovered recently.

Characteristic of Kepler-90i	**Value**
Radius	$1.32R_\oplus$
Semi-major axis	See Part (ix)
Mass	$2.5M_\oplus$
Orbital eccentricity	Approx. 0
Orbital period	14.44912 Earth days
Age	Approx. 2 billion years
Albedo	Approx. 0

(ix) [1 point] Determine the semi-major axis of Kepler-90i about its star Kepler-90.

Solution. By Kepler's third law,

$$T^2 = \frac{4\pi}{GM_{\text{star}}} R^3,$$

from which we obtain

$$R = \sqrt[3]{\frac{T^2 GM_{\text{star}}}{4\pi}} = 1.8488 \times 10^{10}\,\text{m}\,.$$

Note that using either M_{star} or $M_{\text{star}} + M_{\text{planet}}$ will give a similar answer because $M_{\text{planet}} \ll M_{\text{star}}$. Both are accepted.

(x) [1 point] Show that the luminosity of Kepler-90 is approximately $7.28 \times 10^{26}\,\text{W}$.

Solution. Since it is of spectral type G0V, i.e. it is a main-sequence star, the luminosity of Kepler-90 is given by

$$L = 1.2^{3.5} L_{\odot} = 7.28 \times 10^{26}\,\text{W}\,.$$

(xi) [2 points] Show that the surface temperature of Kepler-90i is $929\,\text{K}$.

Solution. By Part (viii), we know that the surface temperature is determined mainly by the solar irradiance on the surface of the planet. Hence, we can attempt to calculate the surface temperature of the planet. (This is because the surface temperature can be primarily determined by radiogenic heat.)

Using the formula from Part (vii),

$$P = \frac{\pi R_p^2}{4\pi d_{S-p}^2} L_{\text{star}}.$$

Here, the subscript p represents the planet.

To then determine the surface temperature of Kepler-90i, we apply the Stefan-Boltzmann law. For a steady state temperature T_p,

$$P = 4\pi\sigma R_p^2 T_p^4.$$

Therefore, by substituting values,

$$T_p = \sqrt[4]{\frac{P}{4\pi\sigma R_p^2}} = 929\,\text{K}\,.$$

(xii) [6 points] With reference to any of the concepts mentioned in the previous parts of this question, the characteristic data of Kepler-90, and its orbiting planet Kepler-90i, discuss the probability of life on Kepler-90i.

In your answer, you should support your argument with reference to relevant calculation(s) and/or theory. You should include at least 2-3 points/arguments (with hopes of scoring more than half of the marks allocated for this question).

Solution. Arguments against:

- This temperature is probably too high for water to exist in liquid state at Earth's atmospheric pressure.
  - An improvement of this argument would be that this calculation assumes the lack of atmosphere. With the presence of an atmosphere, a requirement of habitability to shield against the magnetic field from Kepler-90, this will further increase the temperature of the surface of Kepler-90i. These two factors further support the fact that Kepler-90i would not have a chance of life. (Note: This counts as another argument supported by theory.)
  - Another improvement of this argument that is closer to its star than Earth would be from its own star (as seen from its semi-major axis). For an inner planet, the main determinant of temperature is solar irradiation. Hence, a closer planet would mean that the value of surface temperature obtained would be quite accurate, further supporting the fact that Kepler-90i would not have a chance of life with such a temperature.
- The close distance of Kepler-90i from its star Kepler-90 implies strong solar wind from its host star. If the planet Kepler-90i is rotating at a slow rate, this generates a small protective magnetosphere which cannot prevent the strong solar wind from stripping off the atmosphere. Hence, it might be possible that Kepler-90i is left without its atmosphere and exposed to solar radiation, thus it is unsuitable for life to thrive in.
- We can determine the possible atmospheric composition of Kepler-90i.
  The escape velocity of particles is given by $v_e = \sqrt{\frac{2GM}{R}}$. For simplicity, let us use the root-mean-square speed of the particles $v_{\text{rms}} = \sqrt{\frac{3RT}{M_r}}$.
  A rule of thumb is that the particles must have approx. less than 1/6 of the escape velocity of the particles to be retained in the atmosphere over the age of the solar system (4.5 billion years).
  Hence, for the particles to be retained in the atmosphere, $v_{\text{rms}} < \frac{1}{6}v_e$. Solving for the critical molar mass, $\frac{3RT}{M_r} < \frac{2GM}{36r}$, i.e. $M_r > 23.0\,\text{g}\,\text{mol}^{-1}$.
  This is higher than the Mr of carbon atoms. Thus, carbon/oxygen in its element form OR water molecules ($M_r = \text{g}\,\text{mol}^{-1}$) might not be retained. (Note: Other molecules such as methane that is hypothesised to be essential, also fall out of the required M_r range as calculated.)

Arguments for:

- Though the conditions are extreme, extremophiles might be able to thrive in such extreme conditions. Hence it is not totally impossible that we might not be able to find life on Kepler-90i.
- We can calculate the density of the planet, and see that

$$\frac{\rho_K}{\rho_{K\oplus}} = \left(\frac{M_K}{M_\oplus}\right)\left(\frac{R_\oplus^3}{R_K^3}\right) = 1.14.$$

Hence, the density of Kepler-90i is approximately that of the density of Earth, indicating a terrestrial/rocky origin.

Coupled with the large mass of the planet and the young age of the star system (larger mass than Earth and younger age than our solar system), it is likely the interior of the planet has more than enough internal heat for tectonic and volcanic activities to occur on its surface.

– This contributes to the creation of an atmosphere via out-gassing to protect life from magnetic fields from Kepler-90.
– This acts as a thermostat by negative feedback via the carbonate-silicate cycle (carbon cycle, or any other similar terminologies).
– The ongoing tectonic activities can create continents, and becoming a crucial driver of evolution and biodiversity for complex life on Kepler-90i.

Any other descriptive arguments that are not repetitive, possibly supported additionally with relevant calculations, will be accepted.

Question 3
The Story of a Space Strife Engineer
(19 points)

Disclaimer: The storyline in this question is a work of fiction. Names, characters, businesses, places, events, locales, and incidents are either the products of the author's imagination or used in a fictitious manner. Any resemblance to actual or fictitious persons, living or dead, actual or fictitious events is purely coincidental.

Introduction

A long time ago, in a galaxy far far away, the Galactic Kingdom was on the retreat. The Rebel Army had assassinated Emperor Palpable, and the sympathisers of the Galactic Kingdom were escaping into the Unknown Space.

Among the remnants of the Galactic Kingdom, a new leader emerged – Supreme Leader Snake. As his first edict, a mobile planet that housed a super-weapon capable of destroying star systems was to be constructed. This new military installation was codenamed Starslayer Base.

Eventually a suitable planet was found. Surprisingly, this planet was similar to Earth. During the construction of Starslayer Base, a large hole was drilled from an end of the planet to the other end of the planet along its equator and through the core of the planet. Once drilling was completed, the Weapons Division team was deployed to install the machinery.

Part 1: Not Just LOST, but Very LOST! (Sub-total: 7 points)

You are an engineer of the Weapons Division team. In a moment of folly, you dropped your spanner down the hole that had been drilled through the planet. Knowing that losing your spanner is punishable by death, you immediately boarded the Very Low-Orbit Satellite Transport system (Very LOST) to the other side of the planet. Very LOST is a mass rapid transport system that uses low-orbit satellites instead of trains to transport people across the surface of the planet. This satellite transport system is only orbiting slightly above the surface of the planet, so slightly that it can be assumed that the radius of orbit is the radius of the planet. Surprisingly, this planet has a uniform density, unlike that of Earth.

(i) [1 point] Show that the time it would take for Very LOST to transport you from one side to the planet to the other side of the planet is given by $\tau = \pi\sqrt{\frac{R^3}{GM}}$.

(Note: This is also the time taken for the spanner to fall from the entrance of the hole to the exit of the hole. Thus the Very LOST is able to save your life!)

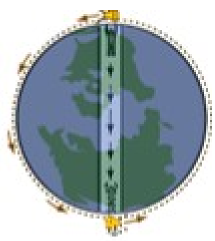


Figure 5: Pictorial representation of the journey of the spanner and Very LOST.

Solution. We want to determine the time taken for Very LOST to travel across half of the circumference of the planet. Using Kepler's Third Law for a circular orbit, we get

$$T^2 = \frac{4\pi^2}{GM}R^3.$$

Hence, $T = 2\pi\sqrt{\frac{R^3}{GM}}$. Therefore, the time τ needed to travel half a revolution is $\tau = \pi\sqrt{\frac{R^3}{GM}}$.

Being the careless engineer that you have always been, this time round, you dropped your spanner while riding the Low-Orbit Satellite Transport system (LOST) while surveying the planet. You gave the spanner enough speed to be out of your reach from LOST, while the spanner continues to orbit at nearly the same orbital radius, r, as LOST. Trying to save your life for the second time, you decide to take a space shuttle to reach an orbital radius of $r + \Delta r$, where Δr refers to a small difference in their orbital radii.

The period of the space shuttle is given by $T + \Delta T$, where T is the orbital period of LOST. The orbital speed of the space shuttle is given by $v - \Delta v$, where v represents the orbital speed of LOST at an orbital radius r.

(ii) [1 point] Show that the speed of LOST, v, at an orbital radius r, is given by

$$v = \sqrt{\frac{GM}{r}},$$

where M is the mass of the planet.

Solution. *Method 1:* Using Kepler's Third Law,

$$T^2 = \frac{4\pi^2}{GM}R^3.$$

Since $v = \frac{2\pi R}{T}$, we have

$$\frac{4\pi^2 R^2}{v^2} = \frac{4\pi^2}{GM}R^3.$$

Making v the subject of the formula yields the desired result.

Method 2: Alternatively, one can use the fact that centripetal force balances with gravitational force, i.e.

$$\frac{mv^2}{r} = \frac{GMm}{r^2}.$$

Making v the subject of the formula yields the desired result.

(iii) [3 points] Show that the difference in period of LOST and your space shuttle ΔT is given by

$$\Delta T = \frac{3\pi\Delta r}{v}.$$

(Hint: Try to obtain an expression that contains $\left(1+\frac{\Delta r}{r}\right)^{\frac{3}{2}}$. Then, since $\Delta r \gg r$, hence $\frac{\Delta r}{r}$ is small. A binomial expansion from the Formula Booklet might help at this point.)

Solution. By Kepler's Third Law,

$$\begin{aligned}
T+\Delta T &= \frac{2\pi}{\sqrt{GM}}(r+\Delta r)^{\frac{3}{2}} \\
&= \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}}\left(1+\frac{\Delta r}{r}\right)^{\frac{3}{2}} \\
&\approx \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}}\left(1+\frac{3\Delta r}{2r}\right) \\
&= T+\frac{3\pi(\Delta r)r^{\frac{3}{2}}}{\sqrt{GM}} \\
&= T+3\pi(\Delta r)\sqrt{\frac{r}{GM}}.
\end{aligned}$$

Since $v=\sqrt{\frac{GM}{r}}$, therefore by substitution we get

$$\Delta T=\frac{3\pi\Delta r}{v}.$$

Note that using the binomial expansion, one can also show that the difference in speed of LOST and your space shuttle Δv is given by

$$\Delta v=\frac{\pi\Delta r}{T}.$$

On the space shuttle, the fear of not being able to catch the spanner with your Super Advanced Magnet (SAM) strikes you. Your plan is to wait for the spanner (which can be assumed to be at the same orbital radius as LOST), your space shuttle, and the planet to form a straight line to activate your SAM. However, being an engineer yourself, you start to worry and attempt to calculate the time at which such an alignment will next occur, if you by any chance happened to miss the upcoming perfect alignment.

(iv) [1 point] Explain why τ satisfies the equation

$$\frac{v\tau}{r}-\frac{(v-\Delta v)\tau}{r+\Delta r}=2\pi.$$

Note: τ is the time required for the perfect alignment. *Solution.* The spanner, being faster than your space shuttle, would have to cover an extra 2π rad to re-establish perfect alignment.

The first term represents an angular distance of $2n\pi+2\pi$ covered by your spanner, and the second term represents an angular distance of $2n\pi$ covered by your space shuttle, where n is an integer. This therefore gives rise to the equation above.

(v) [1 point] The above expression simplifies to $\tau=\frac{2\pi r}{\Delta v+\frac{v\Delta r}{r}}$. By binomial expansion, this can be further simplified to

$$\tau = \frac{T^2}{\Delta T}.$$

What is the significance of this expression?

Solution. Since $T^2 \gg \Delta T$, $\frac{T^2}{\Delta T}$ is very large. Therefore, this perfect alignment probably would only re-occur after a very long time.

Part 2: Stellar Cannibalism (Sub-total: 5 points)

Starslayer Base was designed to consume stars as a power source via a method which drains off all radiation from the surface of the star. Seeking to exact revenge, Supreme Leader Snake ordered that Starslayer Base was to fire upon the Hasnian system where the Rebel Army had recently established the New Republic. Starslayer Base thus fired upon the Hasnian system and completely destroyed all 5 planets within the system. Note that these planets are Earth-like, i.e. we can assume that they each have the mass and radius of the Earth. Star-A1 was fully consumed to supply the energy to fire upon the Hasnian system.

You want to determine a lower bound on the mass of Star-A1 consumed to fire upon the Hasnian system to completely disintegrate the Hasnian system. You may assume Starslayer base has 100% efficiency of converting all the mass of Star-A1 into energy. You may assume that the planets are of uniform density.

(vi) [1 point] If Starslayer Base consumed a star and destroyed the Hasnian system, show that the minimum mass m of the star is

$$m = \frac{3GM^2}{Rc^2},$$

where M and R refer to the mass and radius of the Earth respectively.

(Hint: Use Einstein's mass-energy equivalence.)

Solution. We use Einstein's mass-energy equivalence, $E = mc^2$. This energy is used to completely disintegrate Earth. The energy required to do so is equivalent to the gravitational binding energy of Earth. By the formula in the Formula Booklet,

$$E = -\frac{3GM^2}{5R}.$$

Hence,

$$mc^2 = 5 \times \frac{3GM^2}{5R},$$

i.e. $m = \frac{3GM^2}{Rc^2}$.

(vii) [1 point] Use the above equation to determine the minimum possible mass of Star-A1.

(Note: If you realise that the mass is too low, do not fret. Star Strife is parked under Science Fiction.)

Solution. By direct substitution, we have $m = 1.25 \times 10^{16}$ kg.

The military advisers of Starslayer Base were discussing about habitability around Star-A2, as Star-A2 was the next best candidate star to be used to fire upon the Hasnian system. Habitability was discussed because the Starstrife Base must be positioned within the Goldilocks zone of Star-A2.

(viii) [1 point] Show that the surface temperature of a planet T, and the distance from the star to the planet r, is related by the proportionality relation $T \propto r^{-0.5}$.

Solution. From the Stefan-Boltzmann Law, $P = A\varepsilon\sigma T^4$, where ε is emissivity and σ is the Stefan-Boltzmann constant. Hence, it can be said that $T \propto r^{-0.5}$.

(ix) [1 point] The Goldilocks zone of Star-A2 can be found by considering the zone to be bounded by the melting and freezing point of water at 1 atm. Show that the Goldilocks zone of Star-A2 is approximately 0.60 AU to 1.11 AU.

You may assume that Star-A2 has a surface temperature identical to the Sun. You should give your value as the radius from the centre of the star. Assume that planets in this zone has a similar albedo to Earth, and that the surface temperature of Earth is at 15° C.

Solution. We know that for a star with the luminosity of the Sun, the steady state temperature of an Earth-like planet at 1 AU is 288 K.

Now, the above relation implies that $T\sqrt{r} = k$, where k is a constant. Then we have

$$T_1\sqrt{r_1} = T_2\sqrt{r_2}.$$

With $T_1 = 288\,\text{K}$, $r_1 = 1\,\text{AU}$, we get

- $T_2 = 373\,\text{K} \implies r_2 = 0.60\,\text{AU}$, and
- $T_2 = 273\,\text{K} \implies r_2 = 1.11\,\text{AU}$.

These are the bounds of the Goldilocks zone.

A planet was recently discovered to be located at approximately 0.70 AU from Star-A2. However, upon thorough investigations conducted by the military advisers, they realised that this planet is inhabitable.

(x) [1 point] The military advisors claim you have made an error with your calculations. You disagree; the advisors have not looked at the assumptions of the calculation. But you need to convince them! Explain why this planet is inhabitable even though it lies within the Goldilocks zone.

Solution. The calculation above fails to account for the runaway greenhouse effect. An analogous example would be in the Solar System, where the orbital semi-major axis of Venus is 0.723 AU, but it is not habitable due to the runaway greenhouse effect.

Part 3: Life as an Engineer in an Astronomy Unit (Sub-total: 7 points)

While the military advisers were figuring out the suitability of Star-A2, Kylo Ben had convinced Supreme Leader Snake to check on the suitability of Star-B3. Several astronomers were commissioned to determine certain details about Star-B3. For the surveying process, Starslayer

Base was positioned within the Goldilocks zone of Star-B3, and is confirmed to be habitable (remember, the base is a planet!).

It should be noted that Starslayer Base has a moon orbiting it (by some alien technology). Starslayer Base's moon is similar to Earth's moon in terms of mass and radius. It was also determined that Starslayer Base has a circular orbit about Star-B3 with an orbital radius of 2.0 AU. Starslayer Base's moon has an orbital semi-major axis of 1.4 times the orbital semi-major axis of the Earth's moon about the Earth. The orbital tilt of the moon is 15°.

After your harrowing experience as part of the weapons division team, you successfully requested to be transferred to the astronomy unit. Immediately, you found several tasks waiting for you.

(xi) [1 point] State two conditions for an eclipse to be annular.

Solution.

- Type: Solar Eclipse.
- The angular size of the moon is smaller than the angular size of the star as viewed from the planet.

(xii) [3 points] Determine the eccentricity value of the orbit of Starslayer Base's moon such that the angular size of the moon is the same as the angular size of Star-B3 during aphelion.

(Hint: To prevent you from failing this task, you secretly bought a 'hint' from Kylo Ben. He gave you the following diagram.

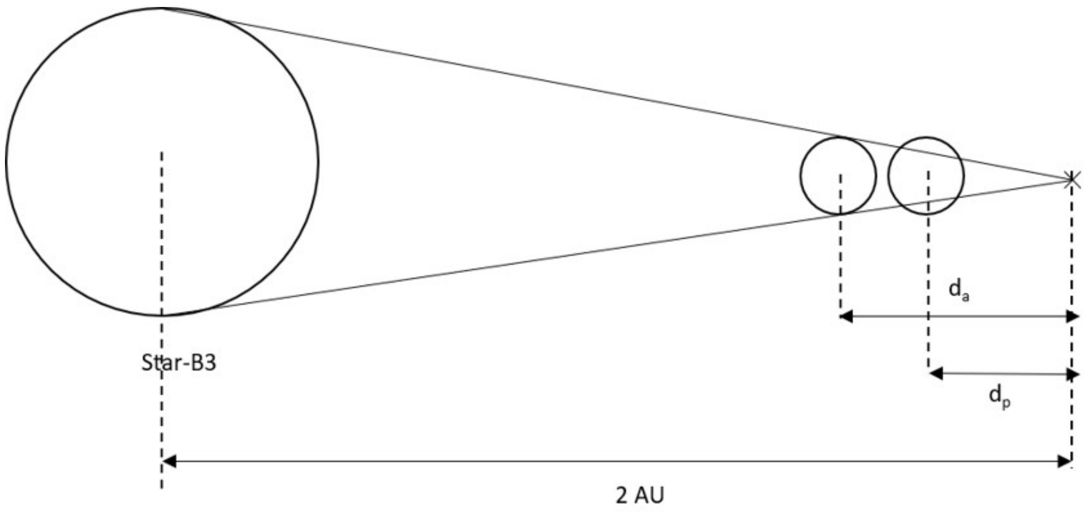


Figure 6: Kylo Ben has good drawing skills.

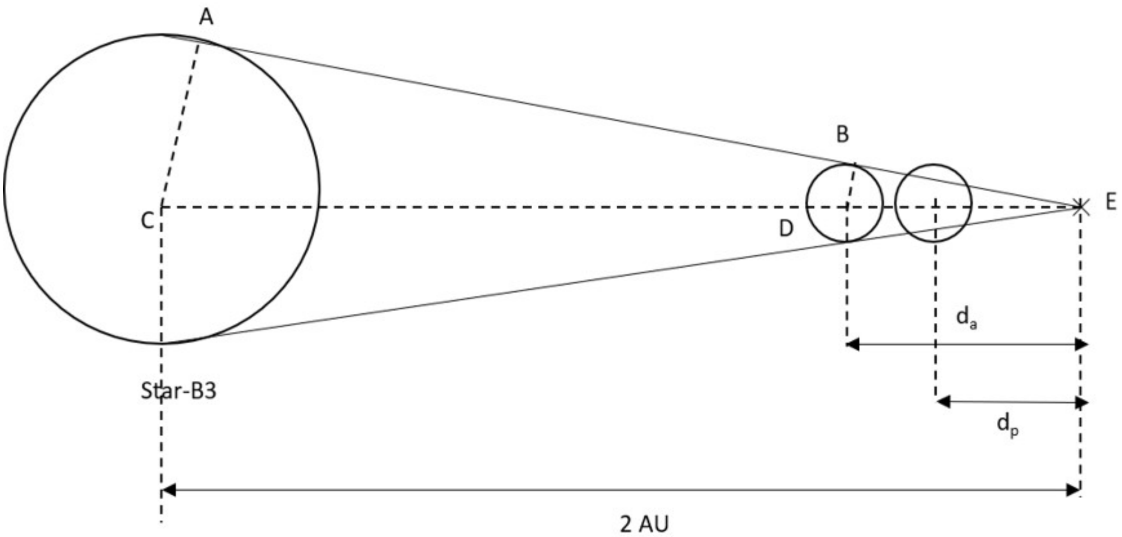
d_a is the distance of Starslayer Base from its moon at aphelion, and d_p is the distance of Starslayer Base from its moon during perihelion. Use geometric arguments to derive your answer.)

Solution. Note that inclination does not affect the range of eccentricity to be determined. Let us determine the critical value of eccentricity. Using Kepler's First Law,

$$d_a = 1.4a_{\text{Moon}}(1+\varepsilon), d_p = 1.4a_{\text{Moon}}(1-\varepsilon),$$

where a_{Moon} is the semi-major axis of the moon.

Consider the following triangle $ABEDC$.



Note that angles CAE and DBE are right angles. By similar triangles,

$$\frac{CA}{DB} = \frac{CE}{DE}.$$

The corresponding physical interpretation of these lengths are

$$\frac{r_{\text{Sun}}}{r_{\text{Moon}}} = \frac{2\,\text{AU}}{d_a}.$$

Solving, $d_a = r_Sun/r_Moon = (2AU)/d_a Solvethisequationtoyielddavalueof2AUxr_Moon/r_Sun =$ $4.992 \times 10^{-3}\,\text{AU} = 7.468 \times 10^{8}\,\text{m}$.

Now, solving for ε using $d_a = 1.4a_{\text{Moon}}(1+\varepsilon)$, we obtain

$$\varepsilon = \frac{d_a}{1.4a_{\text{Moon}}} - 1 = 0.388.$$

(xiii) [1 point] What is the astronomical interpretation of the value calculated in Part (xii)?

Solution. It is the critical eccentricity value that Starslayer Base's moon should have to observe annular eclipse during aphelion. For any values lower than this critical eccentricity value, annular eclipse is not observable.

(xiv) [2 points] Compare the frequency at which solar and lunar eclipses are observed on Starslayer Base to that as observed on Earth. You may assume that the value of eccentricity of Starslayer Base's moon is the same value as calculated in Part (xii).

Solution.

- A larger orbital inclination of this moon's orbit will imply a smaller duration in each eclipse cycle. This would imply that fewer solar/lunar eclipses can be observed per eclipse cycle.

- The orbital period of Starslayer Base's moon is different from that of Earth's moon due to the different lengths of orbital semi-major axis (by Kepler's Third Law).

  Hence, the time interval whereby the moon lies on the line of nodes will be longer from that of the 6 synodic months on Earth. That is to say, the time interval between each eclipse cycle is longer for the system of Starslayer Base and its moon.

Question 4
Qualitative Astronomy and Cosmology
(Total: 20 points)

The following questions are meant to test your understanding of astronomy and cosmology. It is recommended that you answer the following questions in prose to demonstrate your understanding on the subject matter tested, especially for Part 1.

To be awarded the full score, you should discuss how relevant concepts are used and their contextual implications. You may draw labelled diagrams to assist you in your explanations. Marks may be allocated to relevant diagrams. Purely quantitative answers may not be awarded the full score, unless the question is specifically quantitative in nature.

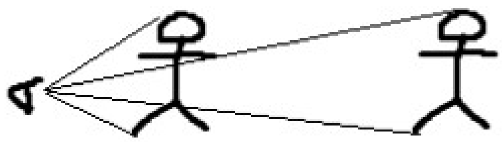
Part 1: Don't Underestimate the Greeks (Sub-total: 12 points)

The Ancient Greeks were around long before Neil Armstrong first landed on the moon, but they were not in the dark as to the vastness of space. By completing the questions below, you should be able to tell that in spite of the minimal information preserved from that era, we know today that the Greeks were aware of certain astronomical facts.

(i) [1 point] The Greeks were aware of the phenomenon that objects in the distance appear to be smaller. Why do objects in the distance appear smaller?

(Note: You are encouraged to draw a diagram to help you explain.)

Solution.



As pictorially described above, for two identical objects, the angular size of the further object is smaller than the angular size of the nearer object. Consequently, the further object will be perceived to be smaller than the nearer object.

(ii) [1 point] The notion that the Earth is round is commonly attributed to Pythagoras of Samos. Give a plausible argument as to how Pythagoras might have known that the Earth was round.

(Hint: Your answer in Part (i) may help.)

Solution. *Argument 1:* Assuming a flat earth theory, as an object moves further from the object, the object would never disappear as it would have to be infinitely far.

This leaves open the possibility for 2 scenarios. A cubic Earth or a spherical Earth. Sailors who sailed beyond the horizon did not "fall off" and in fact returned. This leaves the most plausible geometry for that of Earth to be a sphere.

Argument 2: If Pythagoras was sailing towards a tall mountain, he would have seen the top of the mountain appearing over the horizon before the rest of it. Similarly to the previous argument, if the Earth was flat, Pythagoras would have seen the mountain getting larger proportionally.

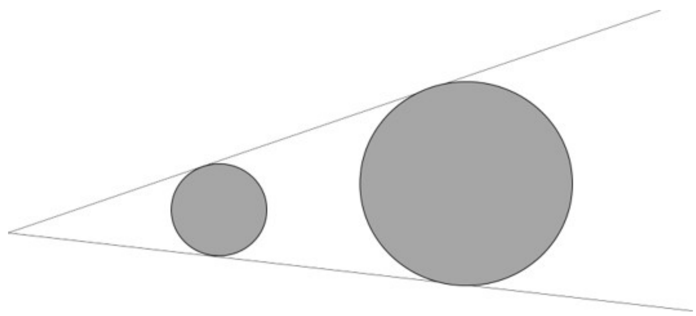
As such, Pythagoras's observation would have been evidence supporting the idea that the Earth is round.

Note: Accept any reasonable explanation that could have happened in 500 BC which would have allowed Pythagoras to deduce that the Earth is round. Partial credit will be given for answers which strongly imply guesswork (e.g. "Pythagoras made a reasonable guess since the moon was round").

Eratosthenes of Cyrene was a Greek astronomer. He is famous for being the first person to calculate the circumference of the Earth.

(iii) [1 point] Eratosthenes knew that the moon was very large. Using your understanding and/or answer to Part (i), explain how Eratosthenes also knew the moon was far away from the Earth.

Solution. On a clear night, Eratosthenes could hold up a medallion that exactly obstructed the moon. The moon must have therefore been significantly further away to have the same angular size.

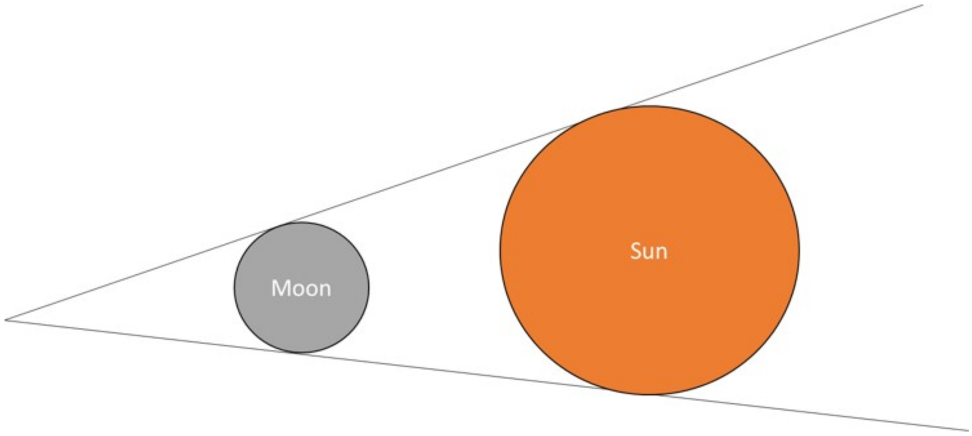


Other acceptable answers:

- Standing on top of a mountain, Greeks were still unable to touch the moon.
- Sailors who travelled past the horizon never reached the moon.
- Acceptable answers are answers that give an indication an object is at a very far distance away.

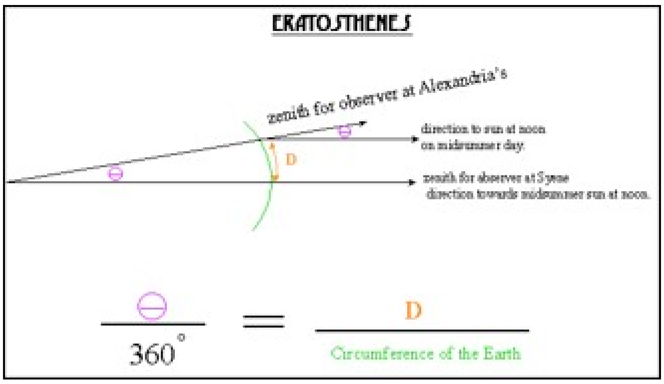
(iv) [1 point] Eratosthenes also knew that the Sun larger than and further than the moon. Using your understanding of Parts (i) and (ii), explain how may have Eratosthenes might have known these two facts.

Solution. During a solar eclipse, the moon is seen passing in front of the sun and obstructing the sun. So the moon is closer to us. Since the Sun has the same angular size as the moon, it must be larger than the moon.



(v) [3 points] Using the knowledge that the sun was very far away from Earth, and the distance between two chosen points on Earth, Eratosthenes was able to calculate the circumference of the Earth. How did he do it?

(Note: You are encouraged to draw a diagram to help you explain.) *Solution.*



First note that since the Sun is very far, the incoming rays are approximately parallel. On the summer solstice at noon at the first point, the sun is at the zenith for an observer at the first point. At this time at a second point, the sun at noon will be off-zenith, at an angle θ. Since the rays are parallel, θ is the angle at the centre of the Earth where the extended zenithal rays meet, therefore

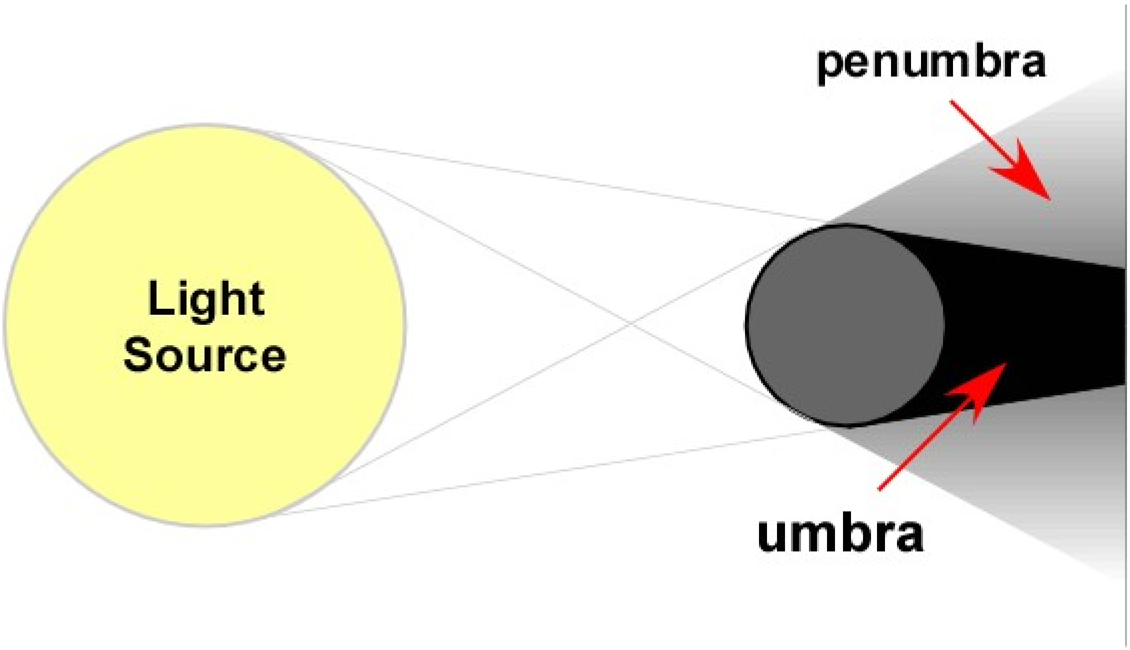
$$\frac{\theta}{360^\circ} = \frac{D}{\text{circumference}},$$

where D is the distance between the two chosen points.

Hipparchus of Nicaea was able to measure the distance of the moon from the Earth using Eratosthenes's value for the circumference of the Earth.

(vi) [1 point] Draw a diagram to show your understanding of the umbra and penumbra.

Solution.



(vii) [3 points] How was Eratosthenes able to obtain the distance of the moon from the Earth?

(Hint: Hipparchus was aware that the shadow of the Earth at the moon's orbit was 2.5 times larger than the moon. How did he know this?

He was also aware the shadow cast by the Earth stretches back 108 times the diameter of Earth. Again, how did he know this?

You are encouraged to draw a diagram to help in your explanation.)

Solution.

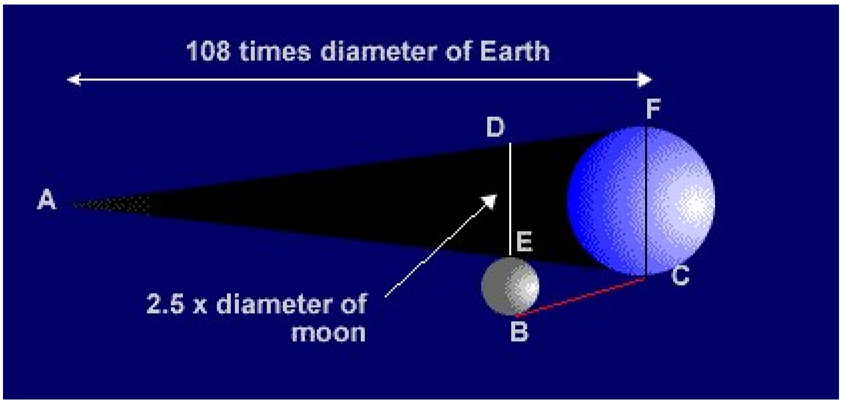


Figure 7: Diagram of what Hipparchus knew.

Imagine holding up a one inch coin at the distance where it just blocks out the sun's rays from one eye. The correct distance is about nine feet away, or 108 inches. If the quarter is further away than that, it is not big enough to block out all the sunlight. If it is closer than 108 inches, it will totally block the sunlight from some small circular area, which gradually increases in size moving towards the quarter. You can also try this with the full moon, which happens to be the same apparent size in the sky as the sun.

Thus, the part of space where the sunlight is totally blocked is conical, with the point 108 inches behind the coin. This is surrounded by a fuzzier area, called the “penumbra”, where the sunlight is partially blocked. The fully shaded area is called the “umbra”.

Imagine now that you are out in space, some distance from the Earth, looking at the Earth’s shadow. The Earth’s shadow must be conical, just like that from the coin. Thus it must be 108 Earth diameters long.

This is because the point of the cone is the furthest point at which the Earth can block all the sunlight, and the ratio of that distance to the diameter is determined by the angular size of the sun being blocked. Hence, the cone is 108 Earth diameters long, the far point being approximately 864000 miles from earth.

Now, during a total lunar eclipse, the moon moves into this cone of darkness. Even when the moon is completely inside the shadow, it can still be dimly seen, because of light scattered by the Earth’s atmosphere. By observing the moon carefully during the eclipse, and seeing how the Earth’s shadow falls on it, Eratosthenes found that the diameter of the Earth’s conical shadow (umbra) at the distance of the moon was about 2.5 times the moon’s own diameter.

It turns out that a century before Hipparchus’s successful determination of the Earth-Moon distance, Aristarchus of Samos had devised a method to measure the Earth-Sun distance. Below is a simplified explanation of his method.

- Aristarchus made use of the moon and its phases. If the Sun was at an infinite distance, then the 1st and 3rd quarter phases would occur when the Moon-Earth line formed a right angle to the Earth-Sun direction.
- If the sun was not at infinity, then the quarter phases would occur at a smaller angle than 90°.

The diagram below provides an exaggerated explanation of the above.

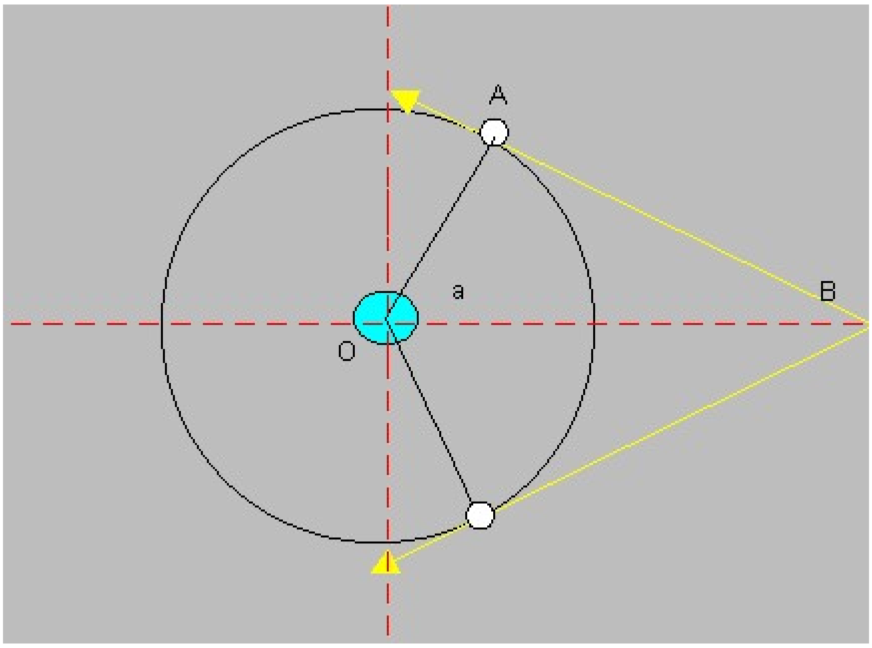


Figure 8: Diagram of Aristarchus’s method.

Using this method, Aristarchus obtained a value of the Earth-Sun distance. Today, we know that the value he obtained was approximately 20 times too small.

(viii) [1 point] Where could Aristarchus have gone wrong?

Solution. Aristarchus's estimate was not conservative enough. He had used an angle a that was not sufficiently close to the actual value.

(Note: This question is meant to test if the students understand that the critical angle in Aristarchus's method depends on the difference between a and $90°$, not how large a is.)

Part 2: Cosmological Primer **(Sub-total: 8 points)**

This section contains questions that are meant to help you understand why certain cosmology theories have gained widespread acceptance. You may answer either qualitatively or quantitatively, wherever relevant.

Fritz Zwicky was a Swiss Astronomer who discovered the Coma cluster. Viewing the Coma cluster today (as Zwicky did back then), we realise that its member galaxies are all moving faster than the expected escape velocity. However, the member galaxies within the Coma cluster are not escaping from the cluster.

(ix) [2 points] Given that the gravitational effects within the Coma cluster can be resolved using Newton's law of gravitation, how might we resolve this apparent paradox? Assume that distance and velocity measurements are accurate.

Solution. By Newton's law of gravitation, the escape velocity of the galaxy cluster is

$$v = \sqrt{\frac{2GM}{R}},$$

where v is the escape velocity, G is the gravitational constant, M is the mass of the galaxy cluster, and R is the distance of the member galaxy in question from the centre of the cluster.

By inspection, the measured value of M must be smaller than the true value.

(Note: This is the first evidence of dark matter.)

Today, the Big Bang model has become the most accepted model to explain the evolution of the universe. The Big Bang model was one of the solutions to Einstein's field equations in general relativity. The following is a solution commonly used to describe the Big Bang model, and is called Friedmann's equation. It defines how the universe derives its expansion.

$$H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2},$$

where

- H is the Hubble parameter today (i.e. the rate of expansion of the universe today),
- K is a value that accounts for the geometry (curvature) of the universe; if $K = 0$ the universe is flat, if $K > 0$ the universe is closed (similar to a sphere), and if $K < 0$ the universe is open (similar to a saddle),

- a is the scale factor (i.e. it describes the relative expansion of the universe), and
- ρ is the total energy density of the universe.

Suppose the universe is flat, i.e. $K = 0$.

(x) [2 points] Modify Friedmann's equation such that ρ is the subject of the equation, and explain the significance of this value of ρ.

Solution. We have

$$H^2 = \frac{8\pi G}{3}\rho,$$

so $\rho = \frac{3H^2}{8\pi G}$.

This is the **critical value** of ρ that represents the value of total energy density of the universe such that the universe is flat.

(xi) [1 point] ρ is a theoretical value and we want to compare it to observational results. Let Ω be the ratio of ρ' to ρ, where ρ' is the observed total energy density of the universe. Express Ω in terms of ρ'.

Solution.

$$\Omega = \frac{\rho'}{\rho} = \frac{8\pi G}{3H^2}\rho'.$$

(xii) [3 points] Ideally, we expect that $\Omega = 1$. However, we have found that $\Omega < 1$. Derive a similar equation for H^2, taking into account the missing energy density.

(Hint: You may want to work backwards from Part (xi), to Part (x), to the equation in the preamble.)

Solution. We have

$$\Omega = \frac{\rho'}{\rho} = \frac{8\pi G}{3H^2} < 1.$$

Hence, $\rho' < \rho$.

To make $\Omega = 1$ as in the ideal case, we need to introduce a term n to represent the missing energy density, satisfying $\rho' + n = \rho$.

Substituting the above result into the equation of Part (x), we get

$$\rho' + n = \frac{3H^2}{8\pi G},$$

i.e.

$$\frac{8\pi G}{3}\rho' + \frac{8\pi G}{3}n = H^2.$$

This was the case for $K = 0$.

For general K, we may reintroduce the K term to get

$$\frac{8\pi G}{3}\rho' - \frac{Kc^2}{a^2} + \frac{8\pi G}{3}n = H^2,$$

since this term takes into account curvature of the universe.

Note: Students are to realise that there needs to be a term in the preamble equation to account for the "missing energy density". The objectives of this question is to help students realise two things.

- Crudely speaking, the "missing energy density" is what we term today as dark energy.
- The term $\frac{8\pi G}{3}n$ is not a new idea, and was originally thought of by Einstein (albeit for a different purpose). Originally, he had included a Λ term in his equations for general relativity – the cosmological constant. The Friedmann equation which includes the Λ term is

$$H^2 = \frac{8\pi G}{3}\rho' - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}.$$

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Question 5
Practical Astronomy
(Total: 20 points)

Part 1: The Night Sky **(Sub-total: 13 points)**

Refer to the image of the night sky at an unknown location on the next page.

(i) On the image, answer the following questions.

(a) [2 points] Identify the cardinal points.

Solution. See solution image.

(b) [1 point] Trace out the 'Little Dipper' and label it accordingly.

Solution. See solution image.

(c) [1 point] Trace the 'Great Square of Pegasus' and label it accordingly.

Solution. See solution image.

(d) [2 points] Trace out two major complete IAU constellations that are visible (other than Cygnus, Ursa Major, and Ursa Minor).

Solution. See solution image.

(e) [2 points] Mark the positions of two prominent nebulae that are visible and label them accordingly.

Solution. List of possible objects: Ring nebula in Lyra, Dumbbell Nebula in Vulpecula, Heart and Soul Nebula in Perseus (accept Merope Nebula but not M45 here), Witch Head Nebula, Flame Nebula, Horsehead Nebula and Great Nebula in Orion and Flaming Star Nebula in Auriga. Accept any other reasonable answers.

(f) [2 points] Mark the positions of two prominent open clusters that are visible (except Hyades) and label them accordingly.

Solution. Double cluster in Perseus, M45 in Taurus, M36, M37 and M38 in Auriga, Cooling Tower or M29 in Cygnus, M35 in Gemini. Accept any other reasonable answers.

(g) [2 points] Mark the approximate positions of two prominent galaxies that are visible and label them accordingly.

Solution. Only Andromeda Galaxy and Triangulum Galaxy.

(*Note: You should detach the image provided and attach it to your answer script.*)

(ii) [1 point] Approximately, what is the latitude of this location?

Solution. Accept answers between 50° N to 70° N. The location used was actually in Khanty-Mensiysk, Russia. It is located at 61° N.

Part 2: Looking at the Moon **(Sub-total: 7 points)**

Tired of looking at deep-sky objects, you decide to view a 3 km wide crater on the Moon with your 50 mm telescope of focal length 600 mm.

(iii) Determine the following quantities if possible. If it is not possible, explain why.

(a) [2 points] The focal ratio of the telescope.

Solution. We have

$$f/\text{ratio} = \frac{f/\text{length}}{aperture} = \frac{600 \times 10^{-3}}{50 \times 10^{-3}} = f/12.$$

(b) [2 points] The magnification of the crater when viewed though the telescope.

Solution. Cannot determine. Focal length of eyepiece used is unknown.

(iv) [3 points] Considering the angular diameter of the crater, can your telescope resolve the crater? Explain your answer. You may assume that light in the visible spectrum has a wavelength of 500 nm and that the Moon is 384400 km away from Earth.

Solution. We have

$$\theta = 2\tan^{-1}\left(\frac{3 \times 10^3}{2(384400 \times 10^3)}\right) = 0.000447157^\circ = 7.80437 \times 10^{-6}\,\text{rad},$$

and

$$R = \sin^{-1}\left(1.220 \times \frac{500 \times 10^{-9}}{50 \times 10^{-3}}\right) = 0.000699^\circ = 1.22 \times 10^{-5}\,\text{rad}\,.$$

Since $\theta < R$, the telescope cannot resolve the crater.

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Detach this page and attach it to your answer script.

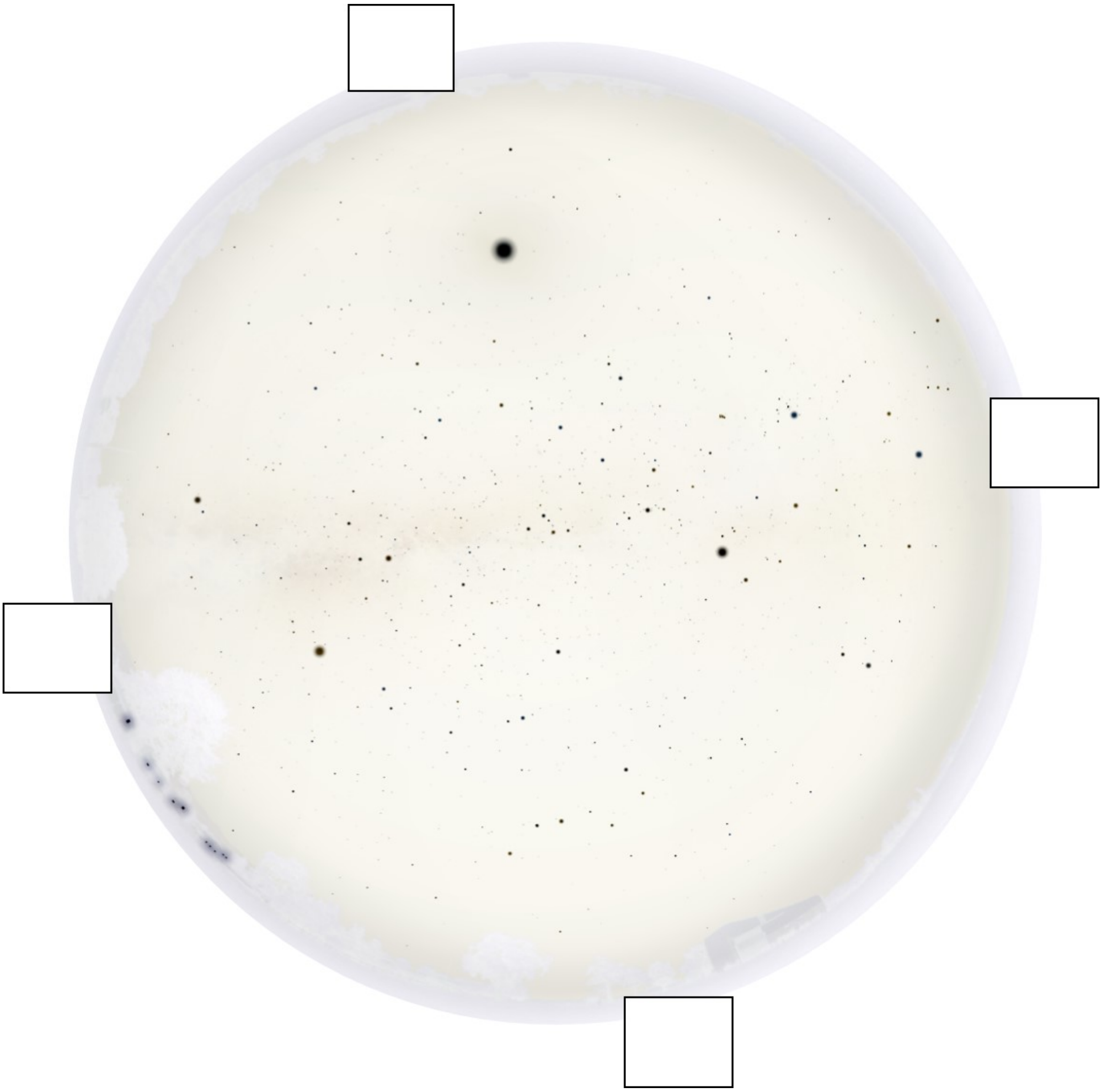

Figure 9: The night sky at some location.

Solution. (For Part 1)

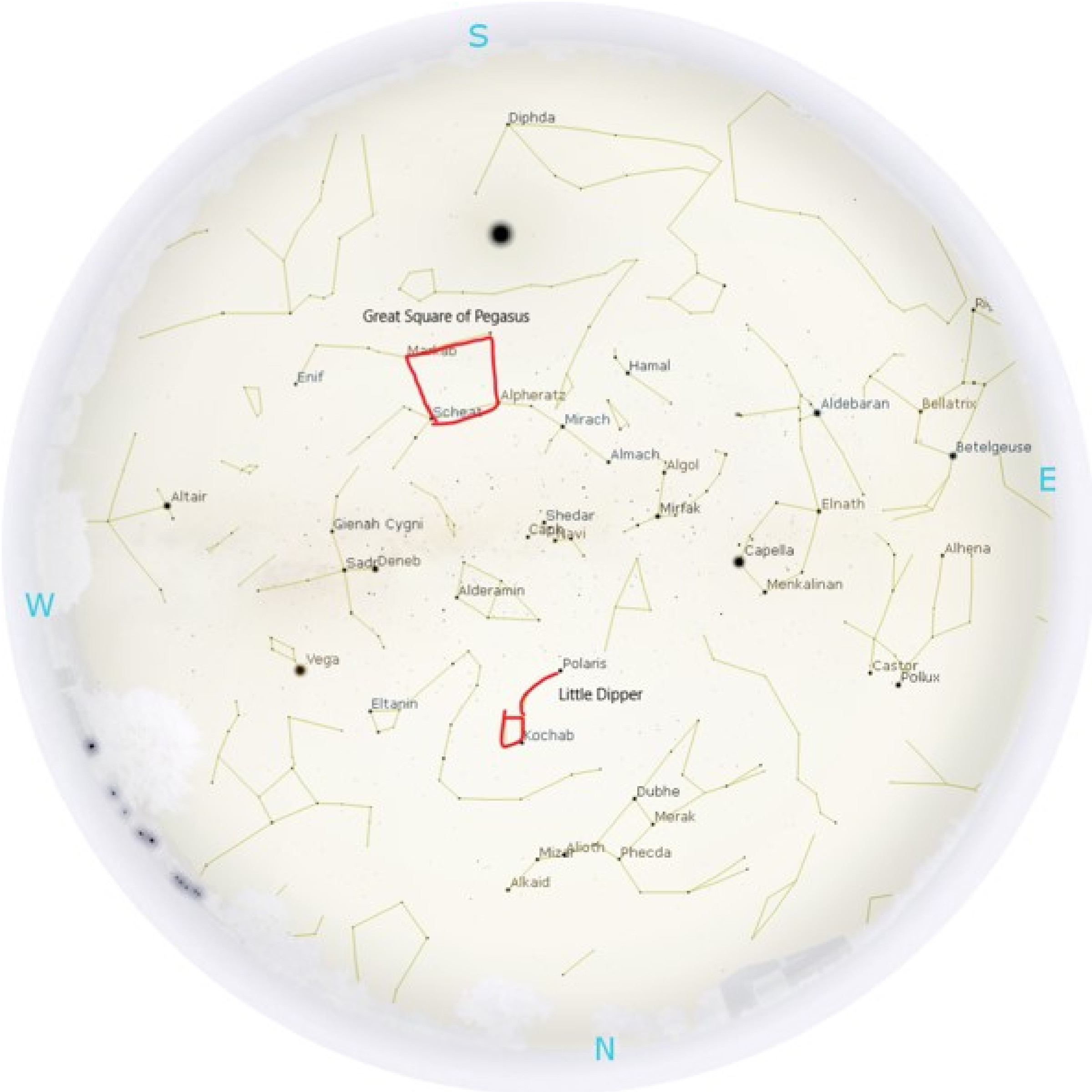

Figure 10: The night sky at Khanty-Mensiysk, Russia, about 7pm.

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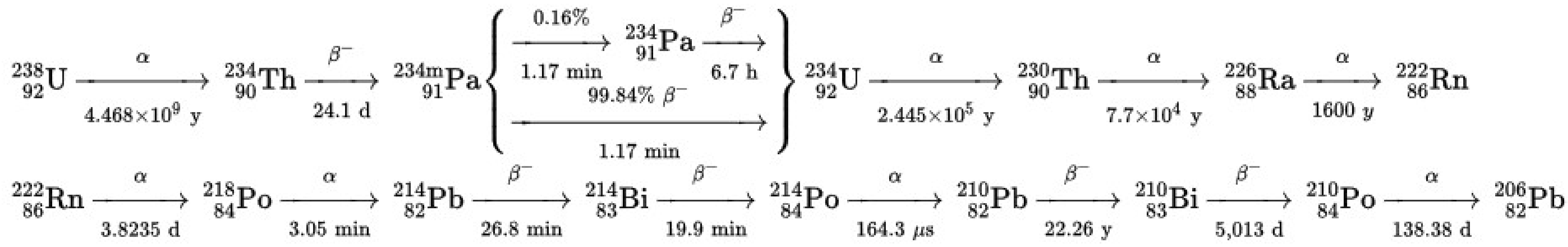
Appendix A

The following is the radioactive decay chain of uranium-238.



Each of the arrows here represents one decay reaction. The nuclear equations for the first three steps are given below.

$$^{238}_{92}\mathrm{U} \to {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He},$$

$$^{234}_{90}\mathrm{Th} \to {}^{234\mathrm{m}}_{91}\mathrm{Pa} + {}^{0}_{-1}\beta,$$

$$^{234\mathrm{m}}_{91}\mathrm{Pa} \to {}^{234}_{92}\mathrm{U} + {}^{0}_{-1}\beta.$$

Note here that we approximate that $^{234\mathrm{m}}_{91}$Pa decays straight into $^{234}_{92}$U.

Example

To determine the energy released from one nuclear reaction, for instance, in the equation

$$^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He},$$

We first calculate the mass defect Δm. This is calculated by taking the total mass of the reacting nuclei, subtracted by the total mass of the product nuclei. For the equation above,

$$\Delta m = 238.051u - 234.044u - 4.002602u = 0.004398u.$$

Note that by the formula booklet, $1u = 1.660539 \times 10^{-27}$ kg. The total energy released is calculated using Einstein's mass-energy equivalence equation

$$E = (\Delta m)c^2.$$

Hence, for the reaction above,

$$E = (0.004398 \times 1.6605 \times 10^{-27})(2.9979 \times 10^8)^2 = 6.564 \times 10^{-13}\,\mathrm{J}.$$

Multi-stage Reactions

For a multi-stage nuclear reaction like

$$^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He},$$
$$^{234}_{90}\mathrm{Th} \rightarrow {}^{234\mathrm{m}}_{91}\mathrm{Pa} + {}^{0}_{-1}\beta,$$

instead of calculating the total energy released in each equation individually, is there a shorter way of calculating the total energy released in this equation without using the mass of $^{234}_{90}\mathrm{Th}$?

Figuring out this "shortcut" will help you in your task for Question 2 Part (iii).