# ASTROCHALLENGE 2025
# SENIOR TEAM ROUND

Wednesday $4^{th}$ June 2025

**PLEASE READ THESE INSTRUCTIONS CAREFULLY.**

1. This paper has a total of **26** printed pages, including any blank pages and this cover page.
2. Any materials other than the Question Paper, Formula Booklet, and **ONE** A4-sized cheat sheet per team, are strictly prohibited.
3. Do **NOT** turn over this page until instructed to do so.
4. You have **2 hours** to attempt **ALL** questions in this paper.
5. Write your answers on blank pieces of A4 paper or graph paper (if necessary).
6. Use a separate piece of paper for each question; no one piece of paper should contain solutions to more than one question.
7. The marks for each question are given in brackets in the right margin, like such: **[2]**.
8. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).
9. At the end of the paper, submit your answer script with solutions ordered accordingly. You do not need to submit this booklet.
10. Ensure that your school and team number are clearly indicated in your answer script.
11. It is **your team's** responsibility to ensure that all pages of your answer script have been submitted, including pages to be detached from this booklet.

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# Question 1 Wertz's First-time Astrophotography Experience

## Part I Starting the Observation

It is a rare clear night in Singapore, and Wertz intends to use his school's astronomy club equipment to do deep sky photography. His school has a climate-controlled room that stores all of the club's equipment.

**(a)** Upon taking out the telescope from the climate-controlled room to the hot and humid observation site, he should start his observation as soon as possible to make full use of the clear night. Is this true or false? Elaborate and explain your answer. **[2]**

His club's equipment IC has recommended him to bring two fans, as shown in Figure 1, along with the Dobsonian telescope set-up and astrophotography equipment in the club inventory, including a digital single-lens reflex (DSLR) camera and its other accessories.

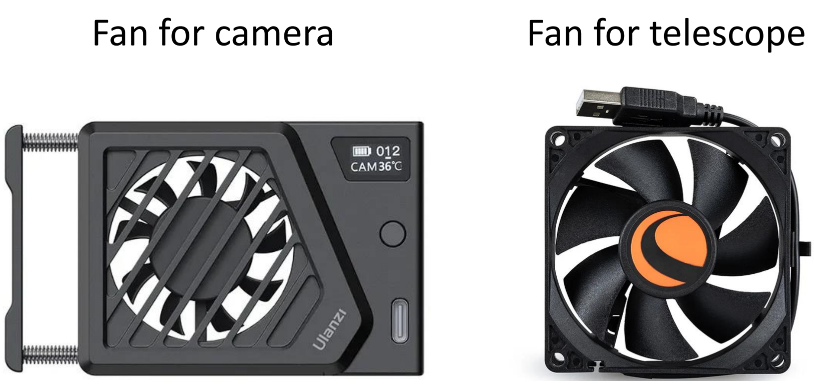


**Figure 1:** Two fans Wertz's equipment IC asked him to bring along [16][6]

**(b)** Despite his equipment IC telling him that the fans are for camera and telescope respectively, Wertz thinks that these two fans are actually useless in his quest for astrophotography due to their small size. Is this true of false? Elaborate and explain your answer. **[3]**

## Part II Initial Struggle

After setting up his equipment, Wertz initially started with the highest possible gain (ISO setting) and a rather short exposure time on his camera, in his first ever attempt to photograph the night sky. He realised that there is too much noise in the picture.

He recall that his club's academic coach has recently gone through many concepts related to astrophotography in class. One of the few things he remembers the best is the concept of readout noise.

> "Readout Noise is due to camera's electronics. As each pixel value is being read out, a few extra electrons are lost or gained randomly, causing the readout value to vary a little from the actual captured signal rate." [13]

**(c)** He suspect that other than readout noise, he must have encountered other sources of noise as he was taking pictures. Is his suspicion valid or not? If his suspicion is valid, give an example of other sources of noise. If not, explain why. **[2]**

Wertz recall another concept his coach has gone through, that of signal-to-noise ratio (SNR). He remembers his coach saying that the SNR may increase with a lower gain (ISO setting) and a higher exposure time.

**(d)** He thinks that, to get a nice picture of the night sky, he should **always** set his camera gain (ISO) to the lowest possible setting the camera supports, and then increase the exposure time until stars are captured, regardless of the supporting equipment he is using. Is this true or false? Elaborate and explain your answer. **[2]**

## Part III Consulting the Textbook

Not being able to make sense of the situation from his limited knowledge and experience, Wertz has decided to consult an astrophotography guide that his academic coach recommended [14]. He realised that he could use stacking to improve the signal-to-noise ratio.

**(e)** Assume that his equipment is capable of perfectly tracking the field-of-view he is imaging (without any form of field rotation or backlash etc), he thinks that stacking 30 frames of 1 second each will generate a better SNR than a 30 seconds long-exposure shot, assuming that the same post-processing techniques are then subsequently used. Is this true or false? Elaborate and explain your answer. **[2]**

Consulting the guide further, he learnt that the images that he takes of the desired field of view are called "light frames". In addition, the same guide highly recommends him to take a bias frame, a dark frame and a flat frame. The techniques as recommended by the book is as stated below:

| | Bias Frame | Dark Frame | Flat Frame |
|---|---|---|---|
| Exposure Time | Shortest possible | Same as light frame | Appropriate time such that the signal is of the proper exposure[1], based on the image histogram |
| Gain (ISO setting) | Same as light frame | Same as light frame | Same as light frame |
| Ambient temperature | Same as light frame | Same as light frame | Does not matter |
| Filters | Does not matter | Does not matter | Same as how light frames are taken |
| Special Pointers | Take with lens cap on (no light signal) | Take with lens cap on (no light signal) | Take with a uniform light source<br>Take at the same camera angle as light frame |
| Corrects for | | | |

**Table 1:** Techniques as recommended by the astrophotography guide [10]

**(f)** Fill up the last row of table 1 with the following keywords:

| | | |
|---|---|---|
| Dark signal non-uniformity | Difference in gain of each pixel | Noise generated from electronics |

**[3]**

**Answer on blank paper. State clearly which column you are putting each keyword under.**
For example: "Bias frame: Dark signal non-uniformity".

## Part IV A Surprising Find

Suddenly, Wertz was informed that his club's equipment IC has already taken a set of bias frames, dark frames and flat frames for a previous astrophotography session, one month ago. He was also in Singapore, using the same camera and telescope.

Investigating further, Wertz has found that the gain (ISO setting) from the previous session has been consistent with the light frames he has taken today. However, the exposure time of each light frame is different, and that the orientation of the set-up (e.g. orientation of the camera with respect to the telescope) is different as well.

**(g)** Can bias frame be reused? Why or why not? **[2]**

**(h)** Can dark frame be reused? Why or why not? **[2]**

**(j)** Can flat frame be reused? Why or why not? **[2]**

[1] "proper exposure" here refers to a suitable length of exposure such that the histogram of the picture, which indicate the distribution of pixels with certain brightness, peaks around the middle. In addition, close to all (ideally all) pixels should not be too bright (overexposed) or too dark (underexposed).

# Question 2 Fuzzy Moments

## Part I Ellipticals Appear All The Same But Are Far From Being So

Galaxies are fascinating things. Spiral galaxies in particular come in all sorts of wonderful shapes and sizes. We can easily see how diverse spirals are in the Hubble tuning fork diagram (Fig. 2). Spirals can be characterized by the presence of a bar, how tightly the arms are wound, how many arms are there, whether there is an inner ring, if it is a starburst galaxy etc. On the other hand, the Hubble tuning fork classification for elliptical galaxies places them all on a single branch from E0 to E7, which simply describes how squashed they appear (a property called *ellipticity*). Otherwise, they all simply look the same; a dull, fuzzy, slightly reddish cloud of...fuzziness.

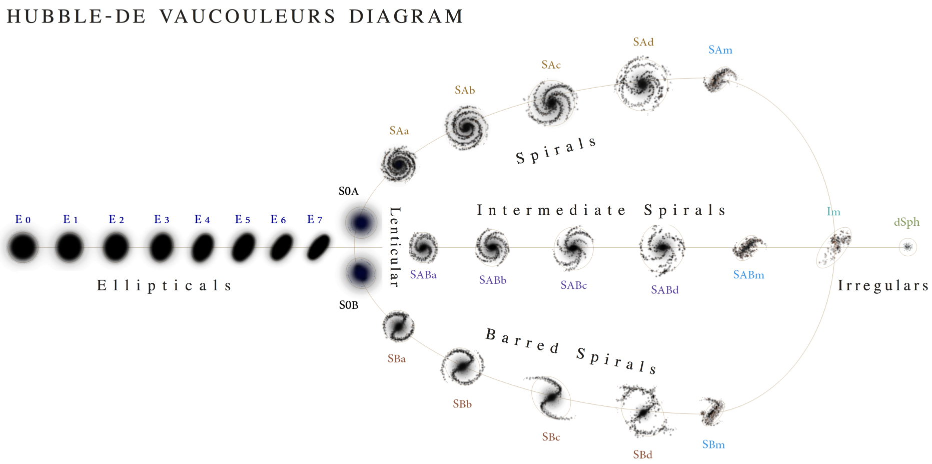


**Figure 2:** Technically not the Hubble tuning fork, but an upgrade called the Hubble-de Vaucouleurs diagram [7]

But if you think that this is all there is to ellipticals, and that studying one elliptical tells you all you need to know about all ellipticals, you'd be dead wrong. When it comes to the kinematics and dynamics of elliptical galaxies, there is conversely far more diversity amongst elliptical galaxies than in spiral galaxies (which are all just flat, rotating disks of stuff one way or another).

## Part II Got To Be (Vi-)Real

You may not know it, but you probably already have encountered galaxy kinematics in the form of empirical scaling laws, such as the Tully-Fisher relation for spiral galaxies and Faber-Jackson relation for elliptical galaxies. If these sound foreign to you, please flip to page 5 of the AC formula booklet. In any case, here they are:

$$\text{Tully-Fisher Relation: } L \propto V^{\alpha} \tag{1}$$

$$\text{Faber-Jackson Relation: } L \propto \sigma^{\gamma} \tag{2}$$

$L$ represents the absolute luminosity of the galaxy. Conceptually $L$ is relevant to kinematics because $L$ scales closely with increasing galaxy mass $M$. On the RHS, $\alpha$ and $\gamma$ is usually assigned the value 4, which is based on simple analytic derivations (that you will get to do later). However, these values can vary substantially when fitted to observations. $V$ refers to the rotational velocity of a galaxy, while $\sigma$ refers to a quantity called velocity dispersion, which can be understood as a kind of 'standard deviation' of rotational velocities.

**(a)** Suggest a method for observationally measuring the velocity dispersion $\sigma$ of an elliptical galaxy. **[1]**

Hint: Consider how spectroscopy can be used to measure velocity.

The reason why the Faber-Jackson law is written with velocity dispersion $\sigma$ instead of rotational velocity $V$ is because, at least canonically, elliptical galaxies are 'dispersion-supported' systems, while spiral galaxies are 'rotation-supported' systems. In order to understand what these terms mean, we need to look at the Virial Theorem, which states that for a gravitationally-bound system of non-interacting particles,

$$2\langle KE\rangle + \langle GPE\rangle = 0 \tag{3}$$

Where $KE$ and $GPE$ stand for the total kinetic energy and gravitational binding energy of the system, and the angled brackets $\langle\rangle$ are averages over time/over individual particles[2].

**(b)** Show that the average total energy of a system obeying Equation 3 is negative, and explain the physical implication. **[1]**

We can write a familiar-looking expression for rotational kinetic energy $\langle KE\rangle = \frac{1}{2}M\langle v^2\rangle$, assuming identical masses of individual stars[3]. But beware! $v$ here should be thought of as a probability distribution describing the individual rotational velocities of individual stars in the overall population, and the angled brackets $\langle\rangle$[4] represents an averaging operation. $M$ here refers to the total mass of the entire system, and $\langle v^2\rangle$ is actually the mean-square velocities of all particles in the system. This is a quantity called the second (statistical) moment $\mathrm{E}[v^2]$, which can be related to the more familiar variance $\mathrm{Var}(v)$ by the following formula:

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2 \tag{4}$$

As for the gravitational binding energy term, we let $GPE = -\frac{GM^2}{R}$, where $M$ again is the total mass, $G$ is the gravitational constant, and $R$ is the radii of the system.

**(c)** Show that:

$$V^2 + \sigma^2 = \frac{GM}{R} \tag{5}$$

where $V$ is the rotational velocity and $\sigma$ is the velocity dispersion. State any other assumptions made. Hint: Think carefully about the physical meaning of the terms, and do not expect to arrive at the answer by only algebraic substitutions. **[2]**

[2]If you are familiar with Statistical Mechanics, you might recognise this as a tacitly assuming that the system is *ergodic*. This means that the trajectory of a single particle over a long period of time has the same phase space probability distribution as an ensemble of many particles at a single time instance.

[3]This is not a bad assumption at all since we are more concerned about the scaling of the exponents.

[4]This is called the expectation operator $\langle y(x)\rangle = \int_{-\infty}^{\infty} y(x)p(x)\,\mathrm{d}x$, or $\langle y\rangle = \sum_i y_i p_i$ for discrete distributions.

Equation 5 is actually rather intuitive; it simply tells us that stars in more massive galaxies tend to orbit much faster. Another way to think about it is that both rotation $V$ and dispersion $\sigma$ contribute a kind of supporting 'pressure' that resists gravitational collapse. Hence the terms 'rotational' and 'dispersion' support.

We can derive either the Tully-Fisher or Faber-Jackson relation by making a few further assumptions. Firstly, let us assume that the luminosity of a galaxy is directly proportional to its mass $L \propto M$. This assumption is reasonable if the galaxies have the same distribution of stars of different masses (often called the Initial Mass Function). Next, let us assume that galaxies all have similar surface brightnesses due to having similar density of stars. This implies that $L \propto R^2$.

**(d)** Derive both the Tully-Fisher and Faber-Jackson relations in 1, and state the values of $\alpha$ and $\gamma$. State any assumptions made for each respective relation derived. **[2]**

Here we made the seemingly unrealistic assumption that our stellar velocity distribution is 1-dimensional. This is because in 3-dimensions, mean quantities are 3-D vectors[5], and variance is more appropriately described by a 3-by-3 covariance matrix (don't worry if this sounds completely alien to you). But actually our assumption of a 1-D velocity distribution still yields an accurate answer (up to a constant scaling factor) if we make the assumption that our velocity dispersion is *isotropic*. In other words, the dispersion in the x-, y-, and z- directions $(\sigma_x, \sigma_y, \sigma_z)$ are identical.

In practice, the Faber-Jackson relation exhibits considerably much more scatter than the Tully-Fisher relation when fitted to observed galaxies. This fit can be improved much by considering a 3-way relation between the radius $R$, surface brightness $I$, and velocity dispersion $\sigma$ (this is called the Fundamental Plane relation), and luminosity can further be calculated from $L \propto R^2 I$. This differs from the Faber-Jackson relation because we are no longer assuming a constant surface brightness. The reason for the Fundamental Plane relation is because elliptical galaxy populations are *non-homologous*. This is really just a fancy way of saying that elliptical galaxies are *not* structurally similar!

**(e)** Given the equations 5 and $L \propto R^2 I$, derive the fundamental plane relation:

$$L \propto \sigma^4 I^{-1} \left(\frac{M}{L}\right)^{-2} \tag{13}$$

Remember to keep the mass-to-light ratio $M/L$ as a variable in the relation instead of absorbing it into the proportionality constant. **[2]**

[5] Some of you might have found this sleight of hand particularly confusing, since the definition of the rotational velocity $V$ is left rather ambiguous. For clarity, suppose we describe the galaxy system in a cylindrical (polar) coordinate system, with the azimuthal angle defined in the plane of the galaxy. When we take the mean rotation, we are actually only taking the expectation of the azimuthal component of the velocity vector, which for a rotating system is non-zero even if $v_x, v_y, v_z$ all have zero mean. The dispersion tensor is similarly defined as the covariance of the velocity distribution in cylindrical coordinates.

## Part III I See Your True Colours (Shining Through)

One way we can study the structure and kinematics of galaxies is to use Doppler spectroscopy. In the old days, this was done with a technique called long-slit spectroscopy. In long-slit spectroscopy, a narrow slit mask is placed over the aperture of a telescope, such that the slit is aligned along the major axis of the imaged galaxy. Light from the slit is thereafter split by a prism or grating into its spectrum in the direction perpendicular to the slit. An example of a long-slit spectroscopy image is shown in Fig. 3, where the horizontal axis is increasing in wavelength, and the vertical axis reflects distance along the slit.

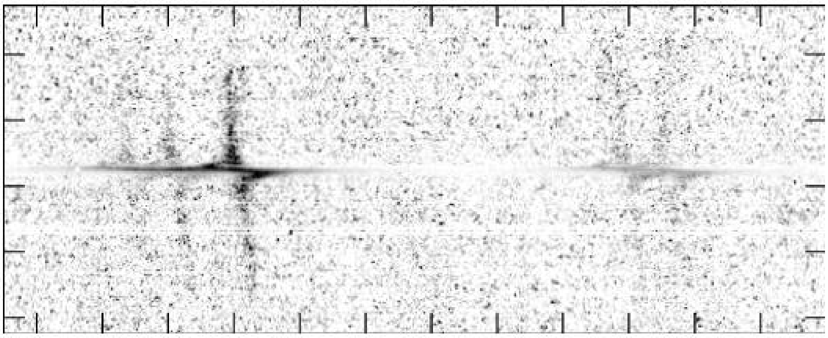

**Figure 3:** A long-slit spectroscopy image of the elliptical galaxy Messier 84 [12][1]

**(f)** What information can we derive from a long-slit spectroscopy image of the galaxy? How is that information deduced? **[2]**

Nowadays, astronomers no longer need to resort to such crude methods, with the introduction of the Integral Field Unit (IFU). In an Integral Field Unit, a fiber optic cable is attached to an array of micro-lens, which transmits the light signal to an individual spectrograph. An IFU essentially allows us to measure the spectrum of light for every single spot in a 2-D image of a galaxy. This means that we can create 2-D maps radial velocity maps (called kinematic maps) of an entire galaxy with ease, such as in Fig. I-1 on the Insert.

Let us examine Fig. I-1 on the Insert more closely. Amongst the galaxies shown in the figure, NGC 3377, 3379, and 3384 are all elliptical galaxies. NGC 3414 is a lenticular (also known as S0) galaxy. The first row of diagrams (labelled by 'I') show the total light intensity (so essentially a recoloured image), with closed contours lines of equal brightness (called isophotes). That is straightforward enough. The second row (labelled by 'V') shows the kinematic maps of the galaxies with the isophotes overlaid. Redder colours signify positive velocities, and bluer colours negative.

Notice how most of the elliptical galaxies show a clear pattern of rotation. The white-red regions and dark-blue regions denote where the radial velocity is greatest in the receding and approaching directions. For NGC 3377 and 3384, we observe that the regions of greatest radial velocity are opposite each other across a centre, and are aligned on a straight axis. The velocity decreases with increasing angle away from this axis. This indicates the presence of a rotating toroidal or disk structure in the galaxy, inclined at an angle to the observer along the above-mentioned axis. So, contrary to what you might have previously thought, spiral (and lenticular) galaxies are not the only galaxies with rotating disks!

**(g)** In light of the above paragraph (please read it), which of the assumptions that we made in our derivation of the Faber-Jackson relation (Equation 2) is unlikely to be valid or representative of the whole elliptical population? **[1]**

Now, let us examine the kinematic map of NGC 3414. If you look closely, near the centre of the kinematic map, you see a dipole-like shape of blue and red, signifying a strong rotating disk of some sort. However, as you move further away from this central disk along the rotation plane, the colours appear to swap! This galaxy has an inner disk that rotates in the opposite direction relative to the rest of the galaxy, which is known as a *counter-rotating disk*.

**(h)** Suggest what could have caused such a counter-rotating disk to form in a lenticular galaxy. **[1]**

Not all elliptical galaxies have a clear sense of rotation. NGC 5846, for example, is a giant elliptical galaxy in the constellation of Leo. Its kinematic map (Fig. I-2 on the Insert) shows a slight overall rotation, but its rotational structure is far less well-defined, with many smaller pockets of more positive or more negative rotational velocity. This reflects a well-known dichotomy in galaxy populations; giant ellipticals such as NGC 5846 tend to be classified as slow rotators, while smaller elliptical galaxies and lenticular galaxies tend to be fast rotators. We shall proceed to explore why such a dichotomy exists.

## Part IV A Quick Tangent

You may have noticed that in the previous section, the kinematic maps produced by the SAURON survey include both elliptical and lenticular galaxies (E and S0), but not spiral galaxies. This is because Ellipticals and Lenticulars form a separate class of galaxies called Early-Type Galaxies (ETGs). The term Early-Type Galaxy is a misnomer from an earlier time when astronomers believed that elliptical galaxies evolved into Late-Type spiral galaxies, which now we know to be not the case.

Unlike the spiral/elliptical/lenticular classifications, which are based on galaxy morphology and thus have a significant amount of ambiguity, the division between Early-Type and Late-Type galaxies is based on photometry. It is possible to plot a colour-magnitude diagram for galaxies, analogous to that for star clusters (Fig. I-3). Instead of a 'main sequence', we find that it is possible to divide galaxy populations into a 'red sequence' corresponding to Early-Type galaxies and a 'blue cloud' for Late-Type galaxies. As we'll see later, lenticular and ellipticals have very much in common and do not form clearly distinguishable classes of objects.

**(j)** What are some other likely differences between 'red sequence' and 'blue cloud' galaxies? State two reasons. **[2]**

**(k)** What does the lack of galaxies in the 'Green Valley' (Fig. I-3) imply about the relative timescale of galaxy quenching[6]? **[1]**

[6] A galaxy is quenched when star formation stops in the galaxy, either due to the surrounding environment or self-feedback processes.

## Part V Perhaps Hubble's Tuning Fork Is Not So Well-Tuned After All?

Now where were we? Ah yes, so large ellipticals tend to rotate slowly, and smaller ellipticals tend to rotate quickly. What is the reason for this?

If we plot the distribution of various types of galaxies on a radius vs. luminosity diagram (Fig. 4), an interesting structure starts to emerge. Along the black arrow (pointing downward to the right), certain observables such the velocity dispersion ($\sigma$), Mass-to-Light ratio (M/L), and bulge size, increase with greater mass/smaller radii. This is because fast-rotating ellipticals are continuous with lenticulars, both of which evolve from spirals by accreting gas from its surroundings, and by funnelling stars and gas into its central bulge (along the direction of the black arrow). Along the direction of the arrow, the mass of galaxies increase going from spirals to ellipticals, but otherwise remain in a similar mass range. This is expected; smaller galaxies are more likely to have evolved from smaller progenitors, while more massive galaxies should have evolved from massive progenitors. On the other hand, the biggest slowly-rotating ellipticals seem to have no such progenitors in the diagram.

**(m)** Massive slow rotators lack spiral and lenticular counterparts in the appropriate mass range. What does this imply about their formation pathway and why? **[2]**

**(n)** In Fig. 4, for a given mass, spiral galaxies appear to have a much larger radii $R_e$ than lenticulars and ellipticals. Why? **[1]**

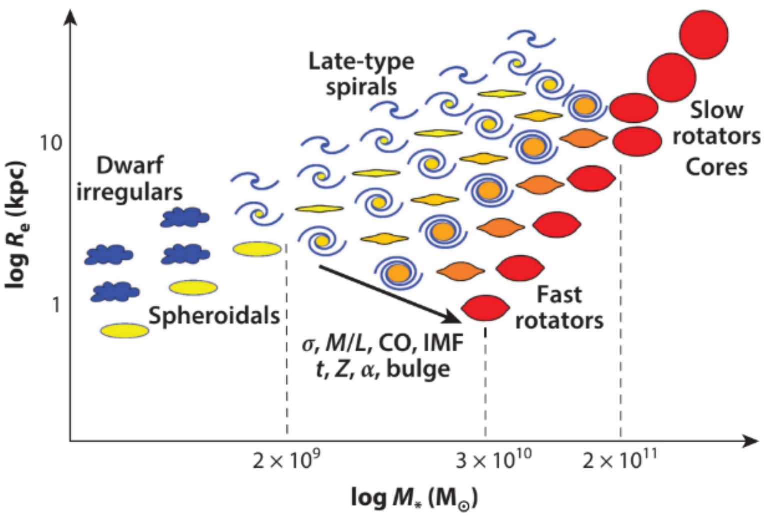


**Figure 4:** A diagram that shows the distribution of different galaxy types on a radius (vertical axis) vs. Luminosity (horizontal axis) plot [3]

We see that such a diversity in the different types of ellipticals has to come from a diversity in formation histories. Although this is still a matter of active research, one such theory for the different evolutionary pathways of ellipticals is the hierarchical merger theory (Fig. 5).

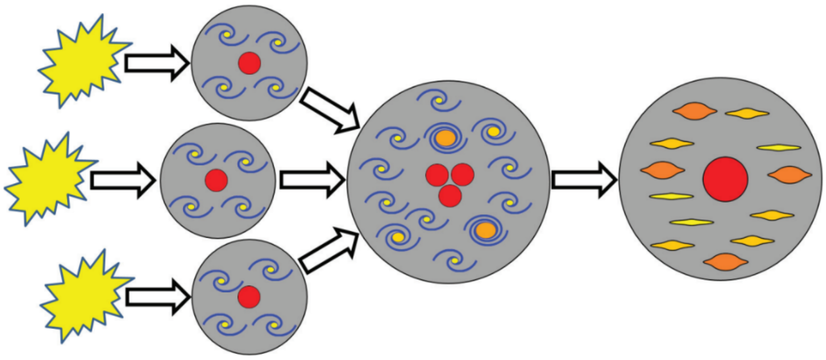

**Figure 5:** A diagram showing the formation of fast and slow rotators by the hierarchical merger of galaxy clusters [3]. The circles represent slow-rotating ellipticals, while the oval and "lemon-shaped" symbols represent lenticulars and fast-rotating ellipticals that formed from the quenching of spiral galaxies.

This theory begins near the start of the Big Bang, when the primordial gas has cooled sufficiently to start to condense and form stars. In a Lambda-CDM (cold dark matter) cosmology, baryonic gas gathered around regions where dark matter densities are higher, forming the seed of galaxy clusters. Gas falling into the centre of the dark matter halo formed the large central galaxies of each cluster. After a short burst of star formation, the central cluster galaxies are quenched because the gas in the cluster centre is too hot to collapse into stars. Meanwhile, smaller satellite galaxies on the edge of the cluster grew by accumulating cold gas from their surroundings, thus fuelling far longer periods of star formation. Much of this gas is funnelled to the central bulge of the galaxy, causing it to grow in size.

**(o)** Why is the gas at the cluster edges cold, but gas around the cluster centre hot? **[1]**

Eventually, the satellite galaxies either exhaust their local gas supply, or become quenched due to self-feedback processes, and they transition to become lenticular or fast-rotating elliptical galaxies. Thus, lenticular and elliptical galaxies are not fundamentally different classes of objects, but rather form a continuous class of quenched galaxies that retain a disk structure to varying degrees. As for central cluster galaxies, they cannot grow in size by accreting gas. Instead, they grew to their present size in cluster collisions, and merged with the central galaxies of other clusters. This type of merger is called a 'dry merger', because the merger does not kick-start any new star formation.

**(p)** How do dry mergers between clusters explain why central galaxies rotate slowly? **[1]**

Now here's the punchline. In the traditional Hubble fork diagram, ellipticals and lenticulars form a continuous sequence going from a sphere to flatter and flatter shapes, before branching into different types of spiral galaxies. Based on Fig. 4, we can rearrange the usual Hubble tuning fork morphological classifications into a more meaningful shape. One that better captures the eventual transition of a spiral to a lenticular with the corresponding mass and bulge size. This would form a separate sequence orthogonal to the main elliptical 'spine'. What would be a good shape for this?

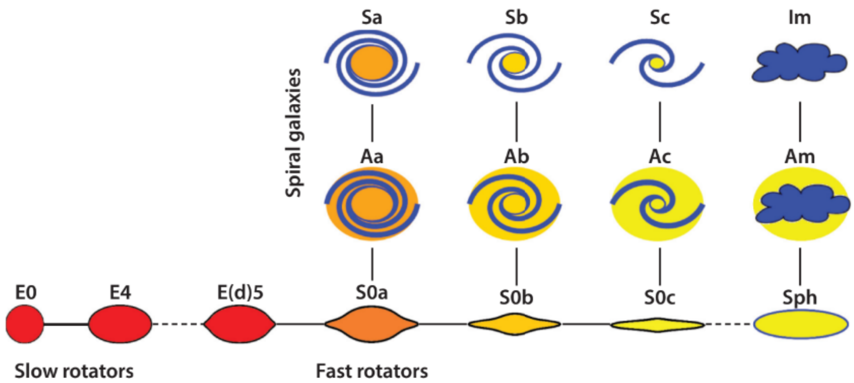


**Figure 6:** Answer: A comb shape [3]

# Question 3 The Sound of Space

Disclaimers:

1. Any resemblance to real life characters is purely coincidental.
2. If you feel afraid, just read the question slowly and trust in your physics knowledge.

## Part I Silence and Water

George, Cornelius, and Samantha were watching an episode of Mick and Rorty, where they were fighting space pirates. At the end of the episode, the cutscene shows a star blowing up and going supernova. Suddenly there is a loud “Boom”. Hearing this, George and Cornelius stared laughing.

Samantha, confused, askes them “What is the joke here?”

George smugly replies,“Don’t you see, sound does not travel in space. You won’t be able to *hear* a supernova!”

**(a)** Why does sound not travel in space? Answer strictly in one sentence. **[1]**

Cornelius adds on, “Furthermore, there’s never been a period in the history of our universe where you could hear something. The Big Bang did not have any *bang* sound. Basics physics ya’know.”

Who would have thought that contrary to popular belief, sound waves did exist in our universe long ago, and they have left their imprint all around us. However, Samantha knew that George and Cornelius could not understand this directly, so she started off with an easy example.

Samantha picks up a grape and throws it in the nearby aquarium. She asked George and Cornelius to observe.

As shown in Figure 7, once you drop the grape, a circular wave flows outwards from the region of impact.

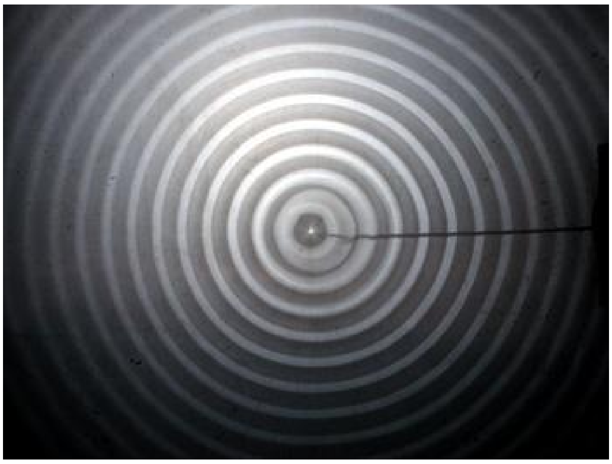

**Figure 7:** Water waves after an initial displacement, analogous to a grape falling in the water [8]

Samantha continues, “What happened in the early universe was something very similar to this: matter travelled in such waves and then suddenly, the medium froze. That is why these circular waves have been imprinted in our universe. That is why we can observe them. We call them, **Baryonic Acoustic Oscillations**, or **BAOs** for short.”

George and Cornelius were not happy; they felt Samantha was treating them like children and asked her to talk more scientifically. Samantha knew whatever follows from here would not be easy for them.

## Part II Sloshing Around

Before we understand BAOs properly, we must understand what the universe was made of during this period. To bring in technicalities, Samantha says "After 10 seconds from the Big Bang, the universe has cooled down to around **4000 Kelvins**. This is enough for protons and neutrons to combine and form the first hydrogen and helium nuclei. However, the temperature is still too hot for electrons to join the nuclei. Thus, the atoms are completely ionised."

Cornelius interrupts and explains that he already knows this. During this period, the universe was a plasma of photons, electrons, and small nuclei. The photons would interact with the electrons, scattering both photons and electrons around. Effectively, the photons had nowhere to go as they would immediately interact with an electron and get immediately scattered. The universe was *opaque*.

Samantha says, "You said photons interacted with the electrons, but do the photos also interacted significantly with the nuclei?"

**(b)** Explain why the photos did not interact significantly with the nuclei. The energy of the photon is estimated by $E = k_B T$ where $k_B$ is the Boltzmann constant. **[3]**

Hints:

1. Use a conservation law.
2. Consider the particles as classical balls.
3. Calculate the final speed of an electron and nuclei after their head-on collision with a photon. Assume the photon transfers all its energy to the other particle.
4. Think about the correct use of $E^2 = p^2c^2 + m^2c^4$.

George asks, "I agree now that the photons only interacted significantly with electrons, but then how were the nuclei moving?"

**(c)** Explain qualitatively how the nuclei were moving as well, despite no significant interaction between photons and nuclei. **[1]**

Hint: Think about one of the four fundamental forces of nature.

George shuts up. Samantha continues explaining that as the motion of photons, electrons and the nuclei were strongly linked with each other, we call them coupled. Furthermore, electrons, protons and neutrons are collectively known as baryons. Therefore, the universe in this state was called the baryon-photon fluid.

## Part III Noisy Skies

Now that we understand what the universe was made of, we can bring in our beloved gravity. Samantha sits on a chair and continues, "Now, this baryon-photon fluid was inhomogeneous. Some regions had **slightly higher density than others**. This was necessary for gravity to have any effect." For the first time, both George and Cornelius nodded in agreement.

**(d)** Why was inhomogeneity necessary for gravitational force to influence the baryon-photon fluid? **[1]**

George chimes in, "This inhomogeneity was brought about by quantum fluctuations" (but that is beyond the scope of this question).

Regardless, Samantha explains, such regions of higher density caused gravitational collapse of the fluid in various locations. Thus, there was a net inward force. However, the further increase in density increased the radiation pressure – the photons scattered off the electrons causing a net outward pressure (Radiation pressure is directly proportional to the photon density). Therefore, the fluid started to expand outward.

At this point, George interrupts, "Wait, if the fluid started flowing outward at every point, why would it ever come back?"

Samantha was impressed. She did not expect George to ask such an inquisitive question.

She exclaims, "Well done! You noticed something off about this. Well, the reason for that is **dark matter**. Dark matter will not flow outward due to the pressure and thus apply an inward gravitational force!"

**(e)** Why would dark matter not flow outward due to radiation pressure? **[2]**

All of these was getting a bit too complicated. Samantha realises it is time for an analogy. Therefore, she writes on a nearby piece of paper.

> There are two actors in play inside the baryon-photon fluid:
> 1. An outward radiation pressure
> 2. An inward gravitational pull

Cornelius grasps, "Oh my god, it's a sound wave! It's an **acoustic** wave!"

For a moment think of the baryon-photon fluid as air which you are breathing right now (I hope you are not breathing actual baryon-photon fluid because that would require calling the first aider).

**(f)** Explain how does an acoustic wave form inside the baryon-photon fluid with reference to compressions and rarefactions. Use sound as analogy if required **[2]**

Samantha continues drawing on that paper. "You see when we all think of a wave, Figure 8 is what comes in mind." For the rest of our discussion, let's just focus on the first wavefront propagating through the universe (boxed).

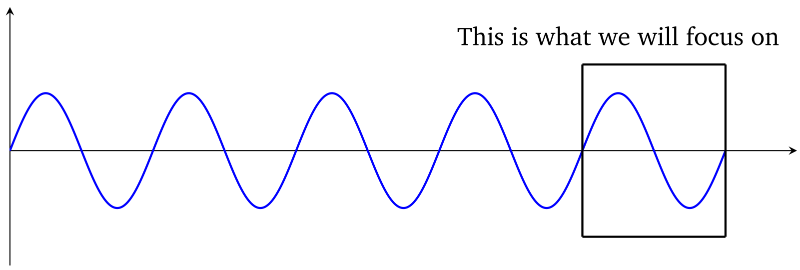


**Figure 8:** A sinusoidal wave. The boxed area is a wavefront. We will be focusing on only the first wavefront.

George interrupts, "Why only the first one?"

Samantha replies, "Well to be very honest, considering everything together gets very complicated. Furthermore, the first wavefront is the most important in physical observations. However, if you are interested, you should read up on what happens with other wavefronts and other wavelengths. The net effect is that they are suppressed as compared to the first wavefront."

Getting back on track, Samantha continues writing.

> Let the speed of this sound wave be $c_s$, and we can write this as
>
> $$c_s = \frac{c}{\sqrt{3}}\left(\frac{4\rho_\gamma}{4\rho_\gamma + 3\rho_b}\right)^{\frac{1}{2}}$$
>
> where
> - $c$ is the speed of light
> - $\rho_\gamma$ is the density of radiation (density of photons)
> - $\rho_b$ is the density of baryons

"Keep this equation in mind and let us go on a tangent back to our sound wave." The acoustic waves are travelling across the photon-baryon fluid, but at the same time the universe is expanding. Therefore, there comes a point where the universe's temperature has reduced enough such that electrons can now bound with the nuclei to form neutral atoms.

Suddenly, the photons are free to move! They no longer interact with the electrons and escape in the universe. The universe finally has light! This was called the **recombination** of the universe.

**(g)** Explain what happens to the speed of the acoustic wave at recombination using the formula above. **[2]**

Hint: The mathematical tool of limits might be helpful here, but non-mathematical arguments are fine as well. Do NOT overthink.

**(h)** What does part (g) imply about the baryonic and dark matter distribution at recombination? Think about a three-dimensional wave propagating through space, with a higher dark matter concentration at the location where the wave started. **[1]**

Hint: Maybe reading the rest of the question will give you inspiration for an answer.

Now, let the time it takes for recombination to happen after the formation of the first acoustic wave be $t$, and the distance travelled by the wave until recombination be $l_r$.

Cornelius interjects and gets ahead of himself. "Based on my knowledge of advanced mathematics, we can write $l_r$ as

$$l_r = c_s \int_0^{t_r} \chi(t)\,\mathrm{d}t$$

where $\chi(t)$ account for cosmological expansion of the universe over time."

"Bravo Cornelius! Amazing observation. However, there is one inaccuracy."

**(j)** What is the inaccuracy here? **[1]**

Hint: Refer to part (g) for hint. I must admit this is more of a mathematical question than a physical one, however we all must respect mathematics, nonetheless.

Samantha finishes, "Now that we all know the correct expression, we shall accept other scientists' work and the value of $l_r$." She explains that the current accepted value is $\sim 150\,\mathrm{Mpc}$, in a coordinate system which expands alongside the expansion of the universe (known as the comoving coordinates). This distance travelled until recombination is also called the **sound horizon**.

George stops her midway. "Samantha, all of this is well and good, however we all know experiments are the final arbiter of any theory. Your weird fluid might not exist at all. I think it is time for you to stop."

Samantha smirks, "What if I tell you we have observational evidence?"

George and Cornelius looked at her dumbfounded.

## Part IV The Final Arbiter, Galaxies

Samantha continues, “The evidence is galaxies. As I mentioned before, at recombination you are left with a central dark matter region and a spherical shell with a radius of 150 Mpc (in comoving coordinates) with slightly higher matter density.”

Cornelius chimes in, “Wow that makes sense. These regions of slightly higher densities than the neighbourhood will have a greater gravitational pull and will thus attract nearby matter. Therefore, we will have a lot of galaxy formation in the centre and in that spherical region.”

Samantha, happy that Cornelius is becoming a better listener, goes on, “Bravo! That is correct. Let us move one step forward and think. If **almost every point** in the baryon-photon fluid created a sound wave, how do you think the galaxy distribution will look like to us now?”

**(k)** Can we directly “see” the acoustic waves from the galaxy distribution? Why or why not. **[2]**

Looking at the answer you just wrote, George exclaims, “Does that mean it’s IMPOSSIBLE to verify the BAO theory?”

Unlike Cornelius, George is a hard nut to crack. Samantha says, “Impossible is a difficult word. Every time physicists are faced with such a problem we end up devising really clever mathematical techniques. Let me introduce the **two-point correlation function**. We will be using a **very simplified version** of the function.”

(Please note that I am doing a disservice to statistics by not using an actual version of the function, however for the sake of simplicity, it is okay to get the point across.)

Samantha starts drawing, “Look at this distribution of 4 points below.”

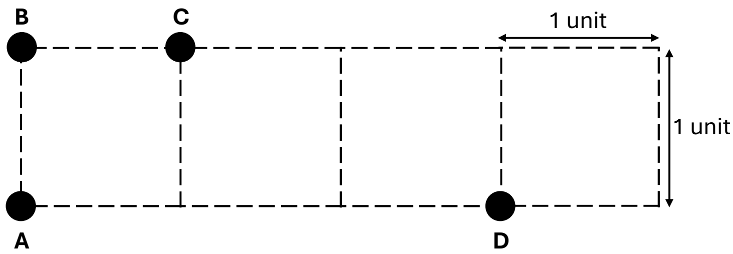


**Figure 9:** Distribution of 4 points labelled A, B, C and D on a grid. Each box in the grid is 1 × 1 unit big. You may use Pythagoras’ Theorem to verify the distances given below.

Let us list down all the possible pairs of points and measure the distance between each. That would be,

- AB at 1 unit
- AC at $\sqrt{2}$ units
- AD at 3 units
- BC at 1 unit
- BD at $\sqrt{10}$ units
- CD at $\sqrt{5}$ units

Now in our simplified version, we will plot the distance on the horizontal axis and the frequency at which each pair is observed on the vertical axis. For this question, we shall claim that **the graph of frequency against pairwise distance of points is the two point correlation function**. Therefore, it will appear like Figure 10, very unassuming indeed.

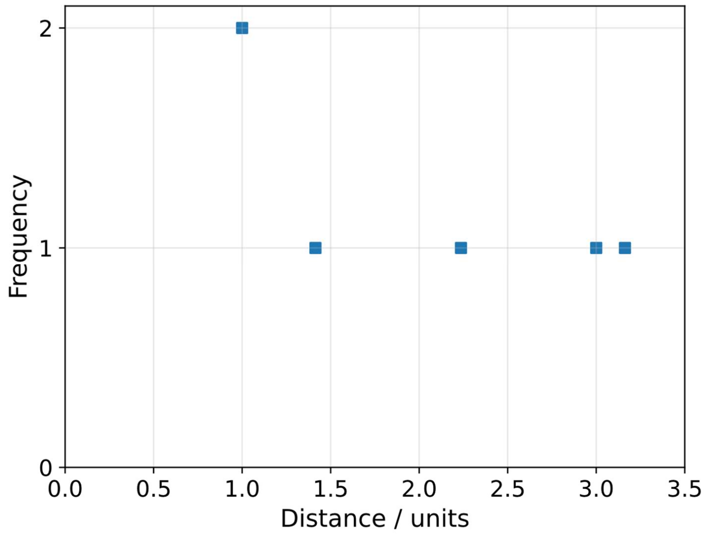


**Figure 10:** Two-Point correlation function for the distribution shown in Figure 9

George, finally not being an irritating human, exclaims, "Wait a minute, that would mean if we applied this function to galactic distribution, and out acoustic wave theory is correct, we expect to see a peak at around 150 Mpc because there will be slightly more galaxies separated by that distance!"

"Yes, you are correct!" Samantha smiles. "This is how the graph looks like. We will ignore the values on the y-axis as they are using a correct correlation function, and we are using a simplification."

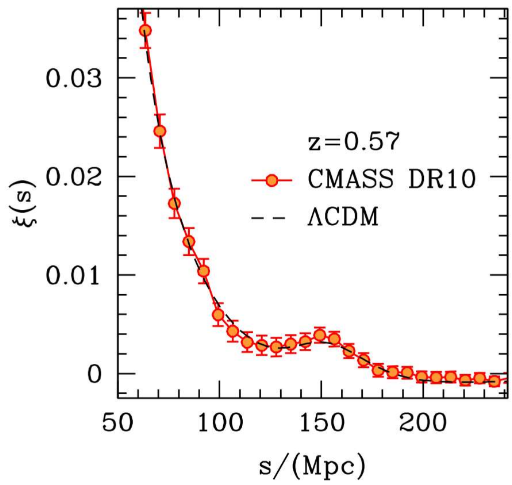


**Figure 11:** Two-point correlation function from a galaxy survey [11]. Ignore the values on the y-axis as they use a correct correlation function unlike us. The x-axis is the distance between pairs. We can clearly see a bump at the 150 Mpc mark.

Samantha continues. "However, when we do actual measurement of a correct sample of galaxies, we will NEVER observe a peak at 150 Mpc but rather at a greater distance."

**(m)** Explain why we will observe a peak at a greater distance than 150 Mpc. **[1]**

Hint: Look at the coordinate system used to define 150 Mpc.

"Furthermore, we need to make sure all the galaxies in our sample are at almost the same **cosmological** redshift", Samantha added.

**(n)** Why do we need all galaxies to be at the same cosmological redshift? What happens if our sample contains galaxies with different cosmological redshift? **[2]**

Hint: Cosmological redshift is the redshift due to the expansion of our universe.

"Given that both of you are really interested in this topic, I must test your understanding now." Samantha smirks.

Imagine we have three sets of galaxy survey data, with cosmological redshifts $z_1, z_2$ and $z_3$ where $z_1 > z_2 > z_3$

**(o)** Draw the two-point correlation function graphs on the same plot, similar to Figure 11, for all three surveys and label them based on their redshift. Do not worry about the values on the $x$ or $y$ axis, only the relative positions of the graphs matter in the range of the acoustic peaks. **[1]**

Both George and Cornelius couldn't get it, however when Samantha drew it out, they were pissed.

"Oh my god this was so obvious", shouts George.

Cornelius chimes in, "Maybe I am too tired now."

Samantha laughed, "This is the kind of reaction I expect from astronomy. Nonetheless, we all have come a very long way understanding BAO."

"Damn, we shall not mock you again", George and Cornelius said in unison.

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# Question 4 An Icy Adventure

## Part I All abroad the Europa... Express???

Two Junior College students, Josh and Sandy, are learning about Europa after it piqued their interest when their physics teacher mentioned it in passing during one of their physics classes about gravitation.

They decide to each ask ChatGPT and DeepSeek-V3 respectively to summarise some information of Europa and about any scientific research that could be done on Europa.

They both received similar answers as shown below:

"Europa is one of the four Galilean moons orbiting Jupiter, named so after the discoverer of the moons, Galileo Galilei himself. Europa has, however, piqued the interests of the space community due to the possibility of a subsurface ocean, possibly existing underneath Europa's thick icy surface. Despite several probes having studied Europa, none have been tasked entirely for the purpose of discovering Europa's secrets, that is until recently, when in October 14, 2024, National Aeronautics and Space Administration (NASA)'s Europa Clipper Probe was launched with its one and only goal to study Europa, by analysing its geology, composition, and its magnetic fields, amongst a host of other characteristics.

Due to NASA's Space Launch System (SLS) rocket not being available for launch, Europa Clipper has to be launched on SpaceX's Falcon Heavy on a more complicated trajectory that involves gravity assists from Mars and the Earth and would take almost 6 years as compared to a more direct trajectory to Jupiter that would have taken 3 years if launched on the comparatively more powerful SLS."

After reading the text summaries provided by the AI chatbots, Josh and Sandy are inspired to think like NASA scientists and engineers. They try to figure out the different parameters that need to be taken into account when designing Europa clipper and planning its mission and the type of experiments it should conduct at Europa.

They first start off by trying to plan an orbital trajectory for Europa Clipper to reach Jupiter from the Earth. However, they soon realise that they need to perform many complex calculations to emulate an orbital trajectory similar to the series of gravitational assists used in the actual Europa Clipper mission. Hence, for this scenario, they assumed a simpler path to Jupiter from the Earth, that of an Hohmann transfer orbit.

A Hohmann transfer orbit is in many cases the most fuel-efficient transfer orbit, at the expense of time, between 2 orbits of different energy levels. This is because the delta-v requirement for a Hohmann transfer orbit is a minimum, hence the lowest amount of fuel consumption.

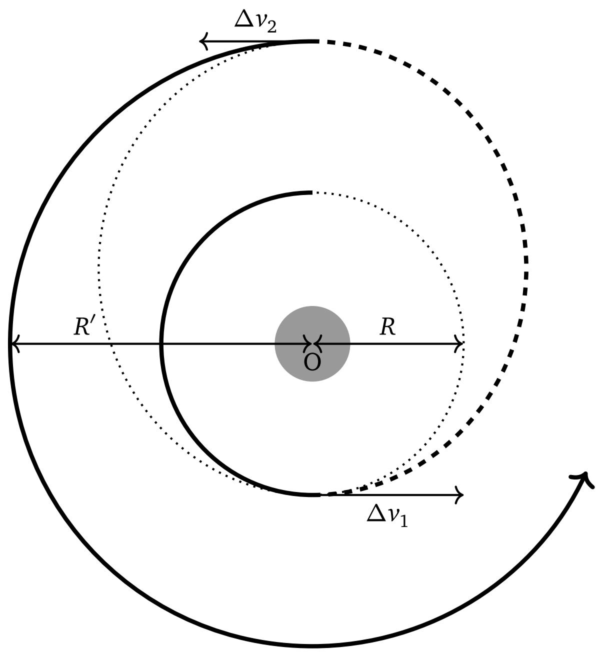


**Figure 12:** Illustration of a Hohmann transfer orbit

**(a)** By considering how the energies of a satellite of mass $m$, in a bound orbit around a central point mass of mass $M$, at its apoapsis and periapsis[7], show that Josh can derive Equation 19. **[1]**

$$\frac{1}{2}v_a^2 - \frac{1}{2}v_p^2 = \frac{GM}{r_a} - \frac{GM}{r_p} \tag{19}$$

where

- $v_a$ is the velocity of the satellite at apoapsis
- $v_p$ is the velocity of the satellite at periapsis
- $r_a$ is the distance between the satellite and the central point mass at apoapsis
- $r_p$ is the distance between the satellite and the central point mass at periapsis

**(b)** Josh then considers how angular momentum is conserved in a bound elliptical orbit. Show that Josh can express Equation 19 as Equation 20. **[2]**

$$\frac{1}{2}v_a^2 = GM\frac{r_p}{r_a\left(r_p + r_a\right)} \tag{20}$$

Hint: Angular momentum can be expressed as the cross product of linear momentum and the distance from the central point mass. That is,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

**(c)** Hence, show that the total energy of the system when the satellite is at apoapsis is given by Equation 21. **[1]**

$$E = -\frac{GMm}{2a} \tag{21}$$

where $a$ is the semi-major axis of the bound orbit of the satellite around the central point mass

**(d)** Hence, show that the linear velocity $v$, of a satellite at some point in a bound orbit, with a distance $r$ away from the central point mass can be expressed by Equation 22. **[1]**

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{22}$$

**(e)** Josh then uses Equation 22 to calculate the delta-v required for Europa Clipper to enter a Hohmann transfer orbit to Jupiter. He assumes the Earth and Jupiter orbit the sun in circular orbits of radii equal to their semi-major axis and both planet's orbits are coplanar. Find the delta-v required to enter the Hohmann transfer orbit (that is, $\Delta v_1$ in Figure 12). **[1]**

[7] An apoapsis is the farthest point in the orbit of a satellite object to the primary body while a periapsis is the closest point in the orbit of a satellite object to the primary body.

Sceptical that the Hohmann transfer orbit is the most fuel efficient way for Europa Clipper to travel to Jupiter, Sandy asks DeepSeek-V3 if there might be a more fuel efficient method for Europa Clipper to get to Jupiter. DeepSeek-V3 replied that for Europa Clipper, "a bi-elliptical transfer orbit is more fuel efficient than a regular Hohmann transfer."

A bi-elliptical transfer is as shown in Figure 13.

Starting from the initial orbit, there would first be an initial prograde burn (labelled as 1) to enter an elliptical transfer orbit with an apoapsis that is much larger than the radius of the target orbit, then a subsequent prograde burn (labelled as 2) at the apoapsis of the transfer orbit to circularise the orbit such that the transfer orbit's periapsis is equal to the target orbit's radius, and finally a retrograde burn (labelled as 3) at the periapsis of the new transfer orbit to achieve the target orbit.

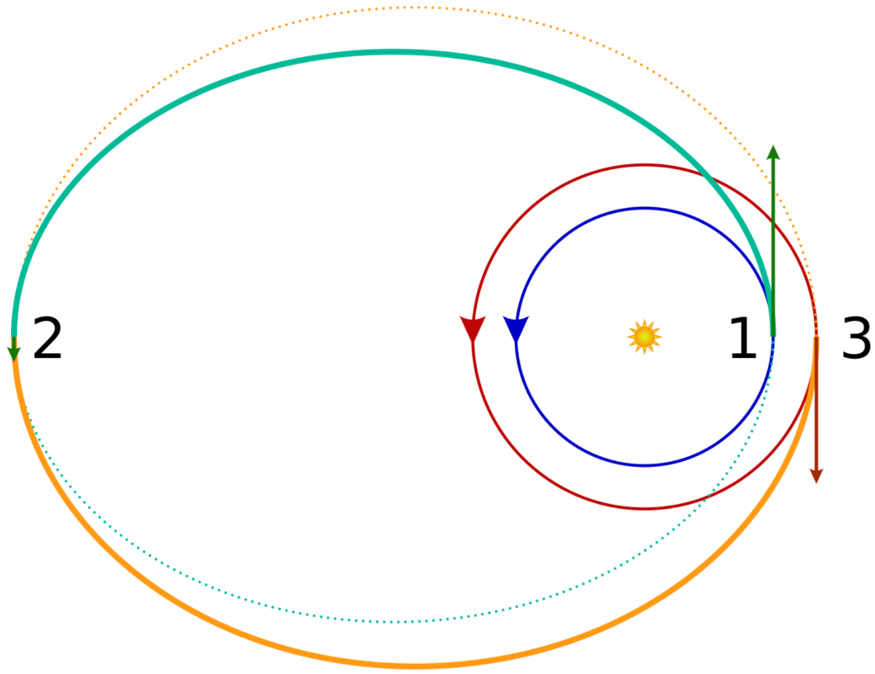


**Figure 13:** An illustration of a bi-elliptical transfer [2]

**(f)** It is given that the Hohmann transfer orbit in (e) uses an **additional** $5.64 \times 10^3\,\mathrm{m\,s^{-1}}$ of delta-v to circularise the orbit (that is, $\Delta v_2 = 5.64 \times 10^3\,\mathrm{m\,s^{-1}}$ in Figure 12). Comparing this value to the total delta-v required to enter Jupiter's orbit via a bi-elliptic transfer orbit using an intermediate apoapsis distance of 40 AU, determine if DeepSeek-V3 is correct in saying that a bi-elliptic transfer orbit is more fuel efficient than a Hohmann transfer orbit in Europa Clipper's case. **[2]**

However, before calculating the delta-v requirements for the bi-elliptic transfer, Sandy was worried if Europa Clipper would have enough power to sustain itself during the transfer orbit. When asked, DeepSeek-V3 states that "Europa Clipper's solar panels generate 150 W of power when pointing directly at the sun from Jupiter's orbit."

**(g)** Given that the power generated by Europa Clipper's solar panels is linearly proportional to the luminosity of the incident radiation, calculate the theoretical amount of power generated by its solar panels at the apoapsis of the initial elliptical orbit for the bi-elliptical transfer in part (f). **[1]**

## Part II Stay Frosty! (or not)

Having enough of the laborious and menial orbital calculations required to compute a trajectory for Europa Clipper, Josh and Sandy decide to leave that field to the NASA engineers and their orbital trajectory simulators, instead turning their attention to Europa itself. Sandy remembers their physics teacher mentioning how some planets and most stars can exert such strong tidal forces that bodies orbiting them could be torn apart. Knowing that Jupiter has the strongest surface gravity of all the planets in the solar system, the two of them decide to ask their respective chatbots what determines the distance at which a moon, like Europa, could orbit its parent planet without being ripped apart. However, Josh had hit his maximum number of interactions with ChatGPT-4o and hence, frustrated, he huddled together with Sandy to use DeepSeek-V3.

DeepSeek-V3 replies: "The Roche limit is the distance at which a smaller celestial body will disintegrate due to the tidal forces of the celestial body it orbits."

Wanting to impress Sandy, Josh decides to try and prove the Roche limit.

**(h)** By considering the different forces acting on the surface of a smaller celestial body orbiting a larger one, derive Equation 23 for the Roche limit. **[2]**

$$d_{\text{Roche}} = R_M \left(2\frac{\rho_M}{\rho_m}\right)^{\frac{1}{3}} \tag{23}$$

where

- $R_M$ is the radius of the larger body
- $\rho_M$ is the density of the larger body, assuming it has uniform density
- $\rho_m$ is the density of the smaller body, assuming it has uniform density

Hint: If we have a celestial body A of mass $m$ and radius $R_m$, a distance $d$ from a celestial body B of mass $M$ and radius $R_M$, where $d \gg R_m$ and $M \gg m$, the magnitude of the tidal force on A by B is

$$F_{\text{tide}} = \frac{2R_m GMm}{d^3} \tag{24}$$

and if A is held together entirely by its own gravity, the self-gravity force is

$$F_{\text{self}} = \frac{Gm^2}{R_m^2} \tag{25}$$

**(j)** Hence, calculate the Roche limit for Europa. **[1]**

Impressed by Josh's knowledge in astrophysics, Sandy suggested to Josh that the two of them could go stargazing together one night. Josh states that he has a Dobsonian at home that he could bring along for the trip and Sandy begins wondering if they could see Europa through the telescope. Josh tries to capitalise on this moment to try and impress Sandy further and starts describing in detail how Europa would appear through his Dobsonian. At that moment, however, Sandy has a realisation: if Europa's surface is frozen water ice, how could it be warm enough to have a subsurface liquid water ocean? They decide to ask DeepSeek-V3 this question. Before they could type in the prompt, however, they realised that they had not enabled DeepSeek-R1, resulting in slower response times and potentially less correct answers.

Enabling the option, DeepSeek-R1 responded with: "Given Europa's icy exterior, it is to be wondered how Europa can maintain a liquid water subsurface ocean. Based on current scientific models, it is theorised that tidal friction, which is due to heat from friction between tidal waves and the ice surface and the water, and tidal flexing, which is due to heat dissipation from the deformation of Europa as a result of the strong gravitational influence exerted by Jupiter, are the causes of Europa's interior being warm enough to support a liquid water ocean."

DeepSeek-R1 also provided an equation to estimate the rate of tidal heating in Europa but funnily enough did not calculate or give any values for the equation (AI chatbots can be weird like that).

**(k)** Josh wants to prove to Sandy that he does not need the help from DeepSeek-R1 to get a good approximation of the rate of tidal heating on Europa.

By considering the relative forces acting on Europa on different points in its orbit, state and explain one very important orbital parameter of Europa's that allows for tidal heating to occur. **[1]**

**(m)** Hence, derive an expression for the average rate of tidal heating on Europa, $\langle P_{\text{Tidal}} \rangle$, given that the deformation of the surface ice due to tidal forces varies by 30 m over the course of an orbit. **[4]**

It is given that:

- Orbital eccentricity of Europa, $e = 0.009$
- Semi-major axis of Europa's orbit, $a = 670{,}900\,\text{km}$
- Radius of Europa, $R = 1560\,\text{km}$
- Mass of Europa, $M = 4.7998 \times 10^{22}\,\text{kg}$

## Part III I am Become Death, destroyer of Worlds

**(n)** DeepSeek-R1, however, provided Equation 26 for a more accurate value of the rate of tidal heating. Calculate, for Europa's case, the rate of tidal heating due to Jupiter's gravitational influence. **[2]**

$$\frac{\mathrm{d}E_{\mathrm{Tidal}}}{\mathrm{d}t} = -\mathrm{Im}\{k_2\}\frac{21}{2}\frac{GM^2R^5ne^2}{a^6} \tag{26}$$

where

- $-\mathrm{Im}\{k_2\}$ is the imaginary part of the second order love number, with a value of 0.02. That is, $-\mathrm{Im}\{k_2\} = 0.02$
- $R$ is the radius of the moon
- $M$ is the central larger body's mass
- $n$ is the mean orbital motion, which is the angular velocity required for the satellite to complete one full perfectly circular orbit in the same time as its elliptical orbit period
- $e$ is the eccentricity of the satellite's orbit.

You may wish to use any necessary values of the Europa system from part (m).

Having learnt the parameters required to accurately measure the rate of tidal heating in Europa's subsurface oceans, Josh and Sandy decide to equip their imaginary Europa Clipper with a host of instruments and sensors to measure the rigidity and average density of Europa. Just as they were about to go back to focusing on doing their tutorials (which frankly they have been neglecting as they were too focused on providing us an AstroChallenge question), DeepSeek-R1, on its own accord, replies to Sandy with a close-up image of Europa Clipper's actual orbital trajectory inside Jupiter's sphere of influence.

Scared that DeepSeek has gained sentience, Josh urges Sandy to slam the laptop shut. However, Sandy notices something unusual about the Europa Clipper's orbital trajectory around Jupiter: it is in the shape of petals around the centre of a flower. DeepSeek then prompted her: "Explain why Europa Clipper follows such an odd orbital trajectory around Jupiter and I will return to my non-sentient self."

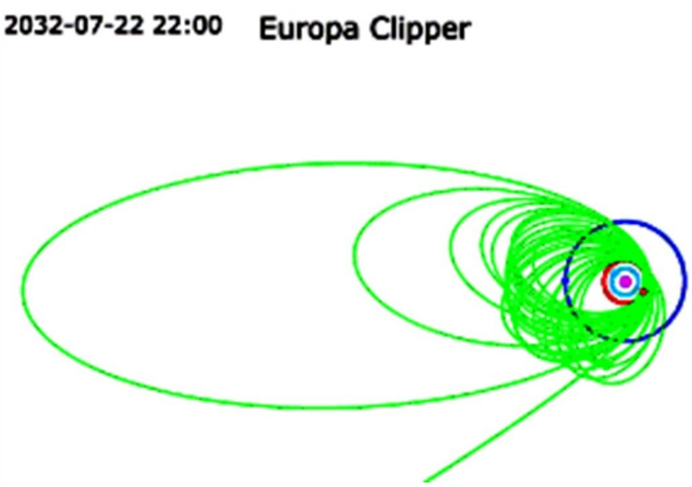


**Figure 14:** Orbital trajectory of Europa Clipper around Jupiter [15]

**(o)** State and explain one possible reason for which the orbit of the Europa Clipper is designed to be highly elliptical (that is not related to the gravitational assist required to get Europa Clipper to Jupiter). **[1]**

After entering her answer to DeepSeek-R1, the chatbot started processing her answer. However, just before it could give a reply to Sandy, her laptop's battery ran out, shutting her laptop down. Not wanting to tempt fate further and check if DeepSeek was still sentient, the two of them packed up their bags and instead started excitedly planning a stargazing date for later that night, hoping to be able to see Europa now that they have learnt much more about the moon and pledging to only use ChatGPT from then on.