AstroChallenge 2021
Senior Team Round

SOLUTIONS

Monday 7th June 2021

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper consists of **33** printed pages, including this cover page.
2. You are required to keep your microphone and camera on at all times throughout the round.
3. You are not allowed to use your keyboard at all times, but you may use your mouse to scroll through the question paper as well as switch to the formula booklet.
4. Any materials other than the Question Paper, Formula Booklet, and **ONE** A4-sized cheat sheet held by **ONE** team member only, are strictly prohibited.
5. You have **2 hours** to attempt all questions in this paper.
6. Write your answers on blank pieces of A4 paper or graph paper. Do **NOT** mix solutions for different questions on the same sheet of paper.
7. You will be given time after the paper to collate your answers. You should collate your answers into **separate PDF files** for each question.
8. It is *your* responsibility to ensure that your answer scripts have been submitted.
9. The marks for each question are given in brackets in the right margin, like such: [**2**].
10. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).

Question 1 Short Answer Questions

Part I Stellar Systems

Consider the following star.

Name	X Astrochallengae
Right Ascension	2h 30min 49s
Declination	$67°28'43''$
Distance	142ly
Apparent Magnitude	4.5
Absolute Magnitude	1.31

Table 1: Information regarding X Astrochallengae.

(a) I currently see this star system[1] at the zenith. Using this information, if able, derive my latitude, longitude as well as the local sidereal time. If you are unable to, you should explain why. **[3]**

Solution:

Being able to observe an object at the zenith implies that it is directly overhead, which suggests that it is also at the prime meridian. From this, we can deduce that the local sidereal time is 2h30min49s.

From the table, the latitude is given to be $67°28'43''$, but the longitude cannot be determined since the local time is not provided.

(b) Without using Table 1, suggest a method to determine the distance to this star system. Explain why your suggested system is appropriate. **[3]**

Solution:

Parallax. The star system is not too far away from us. As such, we can compare it with stars further away (background stars) and obtain an acceptable accuracy.

Partial credit is given for main sequence fitting, since that requires the use of apparent and absolute magnitudes from the table.

(c) You decide to go to Lake Tekapo in New Zealand ($43°53'$S $170°31'$E), a place with very dark skies. However, you find that you never be able to see X Astrochallengae from New Zealand even with a telescope. Suggest a reason why. **[1]**

Solution:

It is below the horizon. The object will never rise because the location is too far south.

(d) Suppose we want to determine if the X Astrochallengae system contains planets. Suggest a method and briefly explain how it works. **[3]**

Solution:

The recommended answer is transit photometry. Dubious answers include direct imaging, radial velocity and astrometry. The star is quite massive, so the planets may not actually be detected using

[1] X Astrochallengae is actually Iota Cassiopeiae.

these methods. However, these answers are also accepted if they are well substantiated.

Part II Dubious Statements

This part comprises 5 statements. For each statement, indicate clearly whether it is **TRUE** or **FALSE**.

Support your answer with no more than 6 sentences, including any assumptions where required. You may draw up to one additional diagram if they aid your explanation.

Mathematical working is not required, and there are no errors in any of the statements below.

Each statement is worth 2 marks, **attributed only to the quality of the justification**.

(e) It is only possible to observe at most two eclipses (either lunar or solar) during a single eclipse season. **[2]**

Solution:

An eclipse season lasts about 35 days. One lunation is approximately 29.5 days. Since the duration of a lunar cycle falls within the time window of an eclipse season, it is possible to experience up to 3 eclipses during an eclipse season (2 lunar and 1 solar, or 2 solar and 1 lunar).

(f) The sky is blue because it reflects the colour of the ocean. **[2]**

Solution:

The Earth's atmosphere scatters shorter wavelengths of light more compared to longer wavelengths. This causes the blue wavelengths of visible light to be more apparent to an observer.

(g) It is possible to observe Venus with the naked eye at any time of the day and night. **[2]**

Solution:

Venus is an inferior planet as it orbits closer to the sun compared to the Earth. This causes Venus to only be observable with the naked eye during the day, early morning, and evening, but not throughout the night.

(h) A white dwarf star that has a mass of 1.5 solar masses ($1.5M_{\odot}$) is considered stable and will eventually turn into a black dwarf. **[2]**

Solution:

The Chandrasekhar limit is approximately $1.44M_{\odot}$. Since this white dwarf is $1.5M_{\odot}$, it is sufficiently massive that electron degeneracy pressure is unable to resist gravitational collapse.

(j) Earth is the only planet in the solar system that experiences aurorae. **[2]**

Solution:

Aurorae are formed when particles from the solar wind are directed toward the northern and southern poles of a celestial body's magnetosphere. There, ionization and excitation ofgases in the atmosphere results in the emission of light of varying colors. As such, aurorae can occur on any celestial body with a magnetosphere and atmosphere.

Question 2 Operation: BINARY

Project: NEUTRON

You are an astronaut working for the United Nations Space Command (UNSC) in the far future. You are currently onboard your ship, the *Pillar of Autumn*, a few AUs away from a binary neutron star system. Your task: analyse the binary star system.

(a) What are neutron stars mainly composed of? **[1]**

Solution:

Neutrons.

Your ship is stationary and in the plane of the orbit of the binary neutron stars. You go to cryo-sleep to allow your ship AI to collect as much data as possible from the binary star system, and once sufficient data has been collected, you are awakened by your ship. Shown below is the graph of the brightness of the binary system against time obtained by your ship.

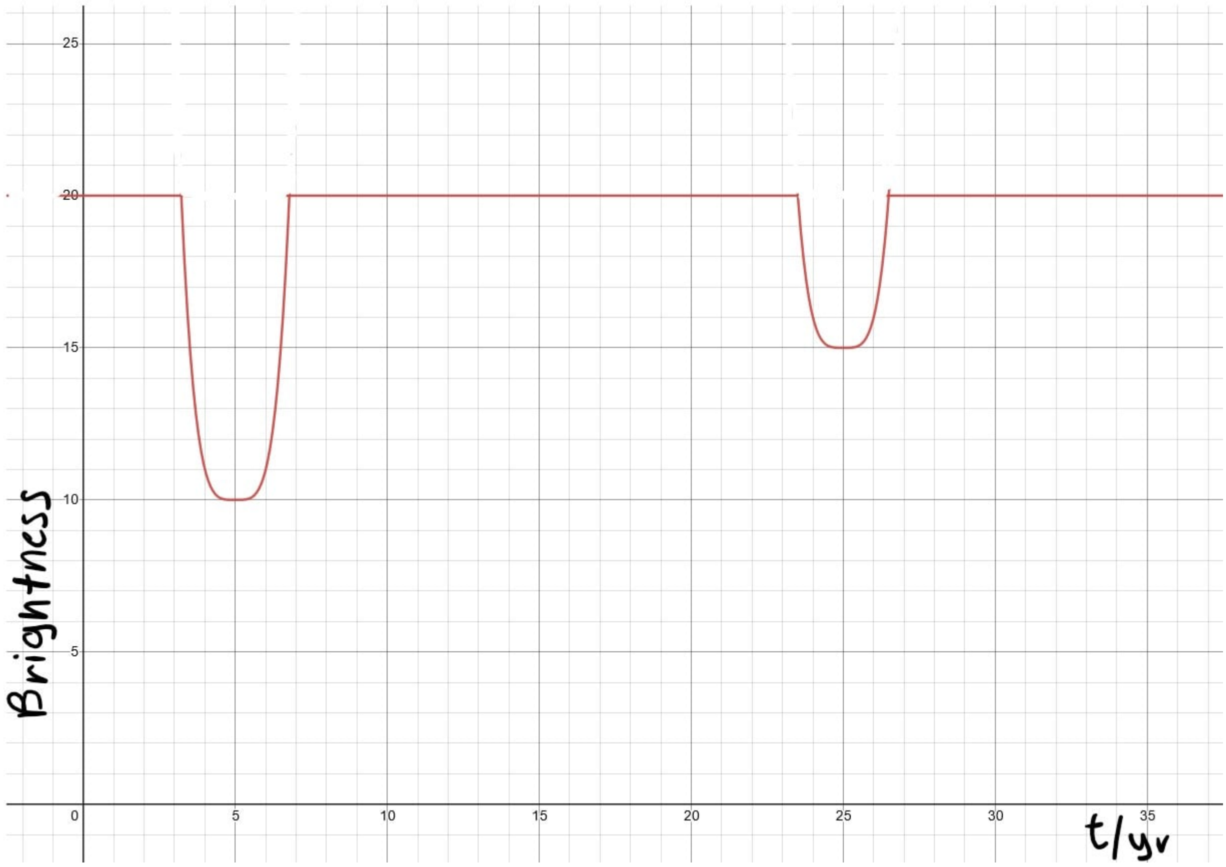


Figure 1: Graph of brightness of the binary star system against time

(b) What is the period of the binary system? **[1]**

Solution:

40 years.

(c) Using Newton's law of gravitation and Newton's second law, show that the period, P, of the binary system is $P = \sqrt{\frac{4\pi^2 R^3}{G(m_1+m_2)}}$. Assume both orbits are circular. [4]

Solution:

We note that by Newton's law of Universal Gravitation, the force on each star is given by

$$F = \frac{Gm_1m_2}{(r_1+r_2)^2}$$

This is the centripetal force, so we can write

$$F = m_1\frac{v_1^2}{r_1}$$

We also know that the velocity is related to the period by

$$v = \frac{2\pi r}{P}$$

This gives

$$\begin{aligned} m_1\frac{v_1^2}{r_1} &= \frac{Gm_1m_2}{(r_1+r_2)^2} \\ \frac{4\pi r_1^2}{P^2} &= G\frac{m_2}{(r_1+r_2)^2} \end{aligned}$$

Since the period of the star is the same,

$$\frac{r_1}{v_1} = \frac{r_2}{v_2}, \frac{r_2}{r_1} = \frac{v_2}{v_1} = \frac{m_1}{m_2}$$

We also have

$$\begin{aligned} R &= r_1 + r_2 \\ &= r_1\left(1+\frac{r_2}{r_1}\right) \\ &= r_1\left(1+\frac{m_1}{m_2}\right) \\ &= \frac{r_1}{m_2}(m_1+m_2) \end{aligned}$$

Substituting our earlier expression gives

$$\begin{aligned} \frac{4\pi^2R^3}{G} &= (m_1+m_2)P^2 \\ P &= \sqrt{\frac{4\pi^2R^3}{G(m_1+m_2)}} \end{aligned}$$

(d) Using Part c, show that the sum of the masses of the binary stars is

$$m_1 + m_2 = \left(\frac{P}{2\pi G}\right)(v_1 + v_2)^3.$$

[2]

Solution:

Using the fact that the period is the same, that is $\frac{r_1}{r_2} = \frac{v_1}{v_2}$, we get

$$r_1 + r_2 = \frac{P}{2\pi}(v_1 + v_2)$$
$$R = \frac{P}{2\pi}(v_1 + v_2)$$

We can substitute this into the expression we obtained in the previous part, $\frac{4\pi^2 R^3}{G} = (m_1 + m_2)P^2$ to get

$$\frac{4\pi^2}{G}\left(\frac{P}{2\pi}(v_1 + v_2)\right)^3 = (m_1 + m_2)P^2$$
$$\left(\frac{P}{2\pi G}\right)(v_1 + v_2)^3 = m_1 + m_2$$

(e) The radial velocity of the two stars have been determined to be $v_{1r} = 20\text{kms}^{-1}$ and $v_{2r} = 40\text{kms}^{-1}$. Determine the masses of the binary stars. [4]

Solution:

We substitute in the values we have into the prior expression to get our value of $m_1 + m_2$, which we obtain as

$$m_1 + m_2 = 6.50 \times 10^{32}\text{kg}$$

We have the relation

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = 2$$

We can solve to obtain the expressions of m_1 and m_2

$$m_1 = 4.334 \times 10^{32}\text{kg}, m_2 = 2.167 \times 10^{32}\text{kg}$$

Project: GRAVITY

Your next task for the UNSC: study gravitational waves generated by a binary black hole system. You will analyse gravitational waves using the wave equation.

(f) In your own words, describe what a gravitational wave is. **[1]**

Solution:

Gravitational waves are disturbances in the curvature of spacetime that are generated by accelerating masses. The waves propagate outwards from their source at the speed of light.

Note: Any other reasonable answers are also accepted.

Like all types of waves, a gravitational wave can be analysed with a wave equation. A typical wave equation has the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}\right).$$

Thankfully, the full strength of the general wave equation is not needed. A simplified version of the wave equation has the form

$$y(x,t) = A\cos(kx - \omega t), \tag{1}$$

where y is the vertical displacement of the wave, x is the horizontal displacement of the wave, t is time, A is the peak amplitude, k is the wavenumber given by $\frac{2\pi}{\lambda}$, and ω is the angular frequency given by $2\pi f$ (where f is the frequency). The vertical displacement of the wave is dependent on two variables, time and the horizontal displacement of the wave.

You will use Equation 1 to model a gravitational wave. Such waves travel at the speed of light, $c = 3\times 10^8 \text{ms}^{-1}$.

Earth is located 1.3×10^9 light years away from the black hole binary system GW150914. Gravitational waves from GW150914 were detected at the newly rebuilt **Laser Interferometer Gravitational-Wave Observatory** (LIGO) located on Earth, a memoir and testament to the historical original.

(g) Take $t = 0$s as the time the gravitational wave was generated and $x = 0$ as the source of the gravitational wave. The gravitational waves are determined to have frequency 150Hz and peak amplitude $A = 2\times 10^{-18}$m. Determine the amplitude of the wave when it is detected by LIGO. Take a year to be 365 days. **[2]**

Solution:

The wave equation is given by

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) \\ &= A\cos\left(\frac{2\pi}{\lambda}x - 2\pi f t\right) \end{aligned}$$

Since this is a wave, we have

$$c = f\lambda \implies \lambda = \frac{c}{f}$$

The rest of the values are given in the question, $t = 1.3\times 10^9$years, $x = 1.3\times 10^9$light years, c is the speed of light, A is the peak amplitude and $f = 150$Hz. We can simply convert all these values to their S.I units and substitute them into the wave equation to obtain the measured value of $y(x,t) = 2\times 10^{-18}$m.

(h) Why are gravitational waves difficult to detect on Earth? **[1]**

Solution:

The amplitude of the gravitational waves when it reaches Earth is very small.

This futuristic version of LIGO is faithful to the original systems used in LIGO of the early 21$^{\text{th}}$ century.

(j) State the method LIGO uses to detect gravitational waves. **[1]**

Solution:

Space-based interferometers.

(k) Explain how the method used in Part j allows LIGO to detect gravitational waves. **[1]**

Solution:

Gravitational waves passing through the interferometer causes disturbances in the local spacetime. Light travelling in the LIGO becomes out of phase and undergo destructive interference, resulting in a measureable signal.

Question 3 All about Limits

Many often know of the limit after which white dwarfs become unstable. But what actually happens inside the white dwarfs and what leads to this instability? We examine the physics behind degeneracy pressure and how this leads to both the stability and instability of white dwarfs and neutron stars. To do this, we model white dwarfs and neutron stars as Fermi gases.

Part I Fermi Gas

Let us first examine a **Fermi gas** in more detail. A Fermi gas is simply a gas of non-interacting fermions. Fermions are particles that include protons, neutrons, and electrons.

(a) In classical physics, what happens to the momentum and the energy of the fermions in a Fermi gas as the temperature T approaches 0? **[1]**

Solution:

As temperature T approaches 0, the momentum and energy of the fermions both approach 0 as well.

Note: Both have to be mentioned for the mark. As T approaches 0, the fermions stop moving classically. Since the momentum and energy of the fermions are both proportional to the velocity of the fermions, they approach 0 as well. The energy of the fermions here is just the kinetic energy since they are non interacting, which means that the potential energy is 0.

Fermions obey the **Pauli exclusion principle**, which states that:

No two fermions can occupy the same quantum state.

Let us examine clearly what this means in two different ways. We know from H2 Chemistry that energy levels are discrete and quantised. The Pauli exclusion principle tells us that there is a limited number of 'slots' that can be filled for each energy level. When these 'slots' are filled up, electrons are forced to occupy higher energy levels.

(b) Explain how this leads to the formation of pressure. **[2]**

Solution:

Higher energy levels correspond to states with higher velocity. Because electrons are forced into these states, there is a higher frequency of collisions and a greater momentum change with each collision, resulting in higher pressure.

Note: Only one of the latter reasons is required for the second mark. Many teams mentioned that the pressure arises because these electrons are forced into these states. While this is not wrong, it does not provide much physical insight into the phenomenon. We do not know how the pressure term changes with the velocity. As a result, only partial credit has been awarded.

Let us now examine this from the momentum perspective. We have learnt from H2 Physics about the Heisenberg uncertainty principle, which gives a relation between the uncertainty Δx in a particle's position and the uncertainty Δp in a particle's momentum. The uncertainty principle is expressed by the following inequality.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

When the fermions are compressed into a very tight space, they have a very well-defined position.

(c) By considering the momenta of the fermions as well as the Heisenberg uncertainty principle, briefly explain how there is formation of pressure. **[2]**

Solution:

The fermions have a very well-defined position, which means that the uncertainty in the position is very small. By the Heisenberg uncertainty principle, there is larger uncertainty in the momentum of the fermions. This means that the average momentum of the fermions are non-zero, which results in pressure when the fermions collide.

Now that we have qualitatively established the mechanism behind the formation of degeneracy pressure due to fermions, we can move on to apply this!

Part II White Dwarfs

In main sequence stars, nuclear processes occurring within the core of the star result in an outwards radiation pressure which balances the inwards gravitational force, giving rise to **hydrostatic equilibrium**. In more compact objects however, the pressure arises due to the degeneracy of fermions (such as protons, neutrons and electrons). Since we are examining white dwarfs, we will specifically be looking at electron degeneracy pressure.

We model the white dwarf as a Fermi gas. In general, we can write the degeneracy pressure due to an **non-relativistic** fermion as

$$P_{\text{fermion}} = knE_F,$$

where k is a constant, E_F is the energy of the fermions, and n is the number density of the fermions, related to the density of the star by the equation

$$n \approx \frac{Z\rho}{Am_p},$$

where Z, A, and m_p are the atomic number, mass number, and mass of the protons respectively. ρ is the density of the Fermi gas. This quantity is important in this discussion.

Given that the momentum of the electrons is given by $p_F = \hbar(3\pi^2 n)^{\frac{1}{3}}$, we can obtain an expression for the electron degeneracy pressure of an electron,

$$P_{\text{electron}} = \frac{k\hbar^2}{m_e}(3\pi^2)^{\frac{2}{3}}\left(\frac{Z}{Am_p}\right)^{\frac{5}{3}}\rho^{\alpha}. \tag{2}$$

(d) Show that $\alpha = \frac{5}{3}$. **[2]**

Solution:

The electron is non-interacting, which means that they do not have any potential energy. This means that their energy is given by the kinetic energy

$$E = \frac{m_e v^2}{2}$$

$$E = \frac{p^2}{2m_e}$$

We can substitute this into our expression for pressure

$$\begin{aligned}
P &= k'nE_F \\
&= k'n\frac{p^2}{2m_e} \\
&= \frac{k'n}{2m_e}\left(\hbar(3\pi^2 n)^{\frac{1}{3}}\right)^2 \\
&= \frac{k'\hbar^2(3\pi^2)^{\frac{2}{3}}}{2m}n^{\frac{5}{3}} \\
&= \frac{k\hbar^2}{m_e}(3\pi^2)^{\frac{2}{3}}\left(\frac{Z}{Am_p}\right)^{\frac{5}{3}}\rho^{\frac{5}{3}} \\
&= \frac{k\hbar^2}{m_e}(3\pi^2)^{\frac{2}{3}}\left(\frac{Z}{Am_p}\right)^{\frac{5}{3}}\rho^{\frac{5}{3}}
\end{aligned}$$

where we have used the fact that $k = \frac{k'}{2}$ and we obtain $\alpha = \frac{5}{3}$, as desired.

Note: The reason for why the energy is given by just the kinetic energy must be substantiated. You might find the assumption that the electrons are non-interacting is very dubious. Considering weakly-interacting electrons instead does not provide much additional physical insight and the mathematics becomes more complicated, which is why we only consider non-interacting electrons here. This is a reasonable assumption in this case as the pressure due to Coulomb replusion is small compared to the pressure due to electron degeneracy.

(e) With reference to Equation 2, explain why, in our discussion, we have considered electron degeneracy pressure but not neutron or proton degeneracy pressure. **[2]**

Solution:

As seen in 2, we can deduce that the degeneracy pressure of a fermion is inversely proportional to its mass. As the mass of the proton and the mass of the neutron is significantly greater than the mass of the electron, proton degeneracy pressure and neutron degeneracy pressure is negligble compared to the electron degeneracy pressure.

Note: Our derivation so far holds for any fermion. Because neutrons and protons are types of fermions, their degeneracy pressure has the same corresponding form.

This outward electron degeneracy pressure balances the inward gravitational pressure. Inward gravitation pressure can be derived using the virial theorem, but we shall spare you the pain of doing so. The equation is

$$p_{\text{grav}} = -\frac{G}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} M^{\frac{2}{3}} \rho^{\frac{4}{3}}.$$

Thus we have relations between the exponents of ρ, and the pressures due to gravity and electron degeneracy.

We are now in a position to look at the stability of the system.

Suppose the white dwarf (or Fermi gas) is at equilibrium and is **non-relativistic**. We give it a small perturbation inwards such that the white dwarf contracts a little. This causes the density of the white dwarf ρ, to increase by a little. This is illustrated in the figure below.

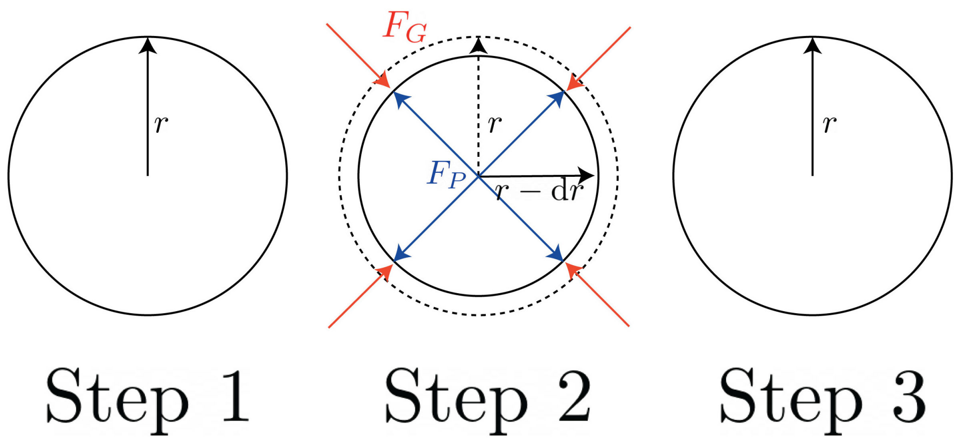


Figure 2: How the equilibrium of the system is restored. **Note that both the arrows representing the forces due to gravity and pressure are there for illustration purposes only and their lengths do not represent the actual magnitudes of the forces.**

(f) This contraction will cause an imbalance in the pressures due to gravity and electron degeneracy. Explain how this imbalance allows the white dwarf to return to its original equilibrium. **[3]**

Solution:

When the white dwarf is given a small contraction, its density increases. This causes both the electron degeneracy pressure and gravitational pressure to increase. However, electron degeneracy pressure increases more than the gravitational pressure because the density's exponent is higher. This means that it is able to act as a restoring force to counteract the contraction, moving back to its equilibrium position and restoring its stability.

Now, let us examine the case for a **relativistic** gas. We now have to use the relation $E_F = p_F c$ to obtain the electron degeneracy pressure for a relativistic electron

$$P_{\text{electron}} = K\rho^{\frac{4}{3}},$$

for some constant K.

(g) By doing a similar analysis to what we have done for the non-relativistic gas, explain why there is an absence of stability. **[2]**

Solution:

A small perturbation inwards causes the density to increase. Since the density exponents of electron degeneracy pressure and gravitational pressure are the same, both forms of pressure increase at the same rate. There is no restoring force to halt its collapse, resulting in instability.

(h) **(BONUS)** Notice that for a relatvisitic Fermi gas, the exponent of the density term in the electron degeneracy pressure is lower than that of a non-relativistic gas. This is counter-intuitive, because the electrons in a relativistic Fermi gas are moving faster. Explain why this is the case. **[1]**

Solution:

Electron degeneracy pressure is a result of the collisions between electrons, which depend on two factors, the frequency of collisions between electrons as well as the momentum change for each collision. Since $E = \gamma mc^2$ and $p = \gamma mv$, an increase in energy or momentum only leads to a very small increase in v since γ depends on v. This means that both factors only increase by a little, resulting in a smaller increase in pressure with an increase in density as compared to the non-relativistic case.

Note: What the question is asking for is the reason why for the same density, relativistic electrons have a lower dependence of ρ as compared to nonrelatvistic electrons.[2]

We can now estimate the mass above which the electrons in a white dwarf will behave relativistically. This occurs when

$$P_F \geq m_e c,$$

which leads to

$$M \geq \left(\frac{1}{2m_p}\right)^2 \frac{3\sqrt{\pi}}{2} \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}.$$

(j) Evaluate this limit in terms of the mass of the Sun, $M_\odot$. What is the name of this limit? **[2]**

Solution:

Subtituting in all the constants in the formula booklet gives us $1.23 M_\odot$. This limit is known as the **Chandrasekhar Limit**.

Hopefully, this has been able to give you more insight behind why a white dwarf behaves the way it does beyond this limit!

[2] A more correct mathematical explanation is linked here.

Part III Neutron Stars

Our analysis above for the pressure, energy, and momentum are for a Fermi gas. Naturally, we expect it to hold for neutrons as well. By using a modified version of the number density for neutrons, we are able to obtain a similar expression for a lower limit of the mass above which neutrons in a neutron star will behave relativistically, given by

$$M \geq \left(\frac{3\sqrt{\pi}}{2m_p^2}\right)\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}.$$

(k) Evaluate this limit in terms of the mass of the Sun, $M_\odot$. **[2]**

Solution:

Substituting in all the constants in the formula booklet gives us $4.93\odot$.

Neutrons have a much shorter de Broglie wavelength than electrons at a given energy, which results in them being spaced much more closely than electrons in a Fermi gas. This means that the pressures within neutron stars are much higher than those of white dwarfs.

(l) Although the actual limit for M is still being improved, it is believed to be around 2 to $3M_\odot$. Suggest a possible explanation for the inaccuracy of the calculated limit in Part k. **[2]**

Solution:

With higher density, the gravitational pressure within the same volume is extremely high. General relativity effects have to be accounted for in such cases due to the high gravity.

OR

A more accurate equation of state should be used. Our analysis assumes that the density is uniform. This is simply not the case in reality.

Note: This is hinted in the paragraph above. Any other reasonable answers are also accepted.

Question 4 A Certain Academic City

In a certain (fictional) academic city[3], much of the activity taking place is devoted to academic studies and research. You have managed to secure a tour through several astronomy-related facilities! First, meet your guides.

Matou	**Kotomi**	**'Akuta'**
		
Unluckiest Student	*Denpa Ojou*	*Smartest One*

"Good morning!" Kotomi[4] greets you brightly.

"H-Hi!" Matou goes next, clearly nervous. "Um...um...you're with me first, and..."

"Get on with it," Akuta scoffs, supremely disinterested. "You guys wanna do buddy-buddy, leave me outta it. Tch."

"We need to be polite!" Kotomi scolds him with a long-suffering air. "And Matou-san[5], calm down!"

It isn't before long that the three start arguing. You wonder if you'll really be okay with them.

[3] Credits: Scenario is shamelessly inspired by the world of Toaru Majutsu no Index. All humanoid character models were created using the avatar creator CHARAT. Koro-sensei (Ansatsu Kyoushitsu) head was self-drawn. Nurufufufu∼

[4] 'Denpa Ojou' translates literally to 'Electromagnetic Young Lady'. Colloquially, it can also mean 'Weird Young Lady'. It's a pun.

[5] '-san' is a polite Japanese honorific. The names are Japanese, because the scenario inspiration is a Japanese light novel/anime series.

Part I A Certain Colourful Index

Your tour starts with Matou's lesson at the city's observatory. Here, you are introduced to his ever-hungry friend. Who for some reason is chomping on his head.

...You wonder if you need the right *filter* to view that particular interaction.

The Colour Index

The *colour index* of an astronomical object is a numerical value that indicates its colour. It is a fairly common occurrence in plots involving colour – for example, most modern Hertzsprung-Russell diagrams plot absolute visual magnitude against the B–V colour index.

To understand how the numerical value is derived, we first need to understand the concept of a filter. A *filter* restricts light observed to a certain range or band of wavelengths, and is named for the wavelengths allowed through. For example, a blue filter allows a band of wavelengths associated with blue light, and *only* this band of wavelengths, through.

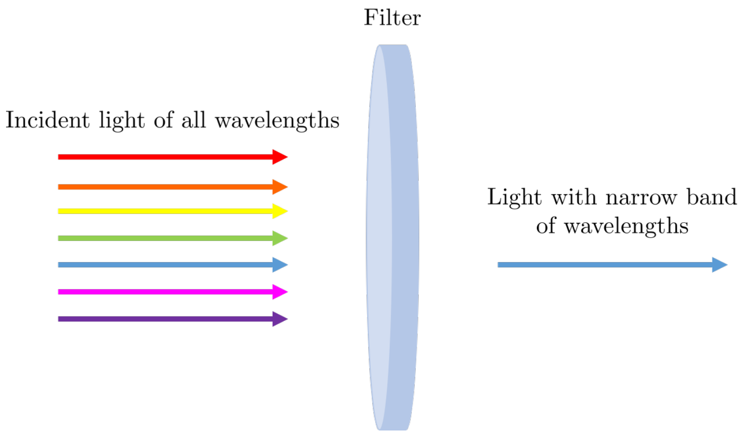


Figure 3: A filter blocks all wavelengths except a narrow band.

In optical astronomy, three filters are the most commonly used – **U**ltraviolet, **B**lue, and **V**isual – forming the UBV system. Sometimes, two additional filters – **R**ed and **I**nfrared – are used, forming the UBVRI system. To determine a colour index, we must specify a choice of two filters to use. For example, if we specified the filters B and V, we obtain the B–V colour index. The B–V colour index of an astronomical object is

$$\text{B–V colour index } = m_{\text{B}} - m_{\text{V}},$$

where m_{B} is the apparent magnitude of the object through the B filter, and m_{V} is the apparent magnitude of the object through the V filter. Analogous computations hold for other colour indices.

(a) True to his unlucky title, Matou is asked the first question.

"Under ideal conditions, knowing **only** the colour index is enough to determine colour, an intrinsic property. And this is because the colour index is a quantity independent of distance. But the colour index is an apparent magnitude difference, and apparent magnitudes depend on distance. So why doesn't the colour index?"

As the star (pun completely intended) visiting student, help him out by answering the question. **[2]**

Solution:

Solution 1:

The first key observation is that since magnitudes depend logarithmically on brightness, a magnitude difference can be transformed into a brightness ratio

$$m_1 - m_2 = k \log \frac{B_2}{B_1},$$

where k is some constant, B_1 and B_2 are brightness corresponding to m_1 and m_2 respectively.

The second key observation is the brightness equation $B = \frac{L}{4\pi d}$. This gives $\frac{B_2}{B_1} = \frac{L_2}{L_1}$, a luminosity ratio, which is clearly independent of distance. Hence $m_1 - m_2$ is also an intrinsic value and is independent of distance, and so it determines stellar colour.

Solution 2:

Observe by the distance modulus relation that

$$m - M = 5 \log \left(\frac{d}{10\,\text{pc}} \right).$$

It follows then that

$$m_B - m_V = 5 \log \left(\frac{d}{10\,\text{pc}} \right) + M_B - 5 \log \left(\frac{d}{10\,\text{pc}} \right) - M_V = M_B - M_V.$$

Observe now that M_B and M_V are absolute magnitudes in the B and V values respectively. Clearly these are intrinsic properties, and therefore the colour index does not rely on distance.

Note: The difficulty here is to somehow get rid of the 'distance' part. The key is to recognise the 'brightness version' of the relationship between luminosity and absolute magnitude, when distances are factored in. The second step is then to realise that the brightness ratio is simply a luminosity ratio, which is an intrinsic property.

Even More Index!

Under ideal assumptions, a single colour index suffices to determine colour. However, reality is often disappointing. Various factors (e.g. interstellar extinction) can affect the colour index value, and different colour indices have different sensitivities to various such phenomena. It is for this reason that multiple colour indices are used.

Star	V apparent magnitude	B–V index	V–R index	R–I index
He 767	10.69	0.62	0.25	0.26
He 601	11.43	0.73	0.32	0.27
AP 25	12.25	0.88	0.41	0.34
AP 55	13.91	1.11	0.56	0.41
AP 86	14.31	1.32	0.81	0.60
AP 20	15.66	1.55	1.28	1.07
AP 60	15.82	1.70	1.34	1.13

Table 2: Photometry data of some low-mass stars in the α Persei cluster, approximately 570 ly from Earth. Edited from Stauffer et al. (1985).

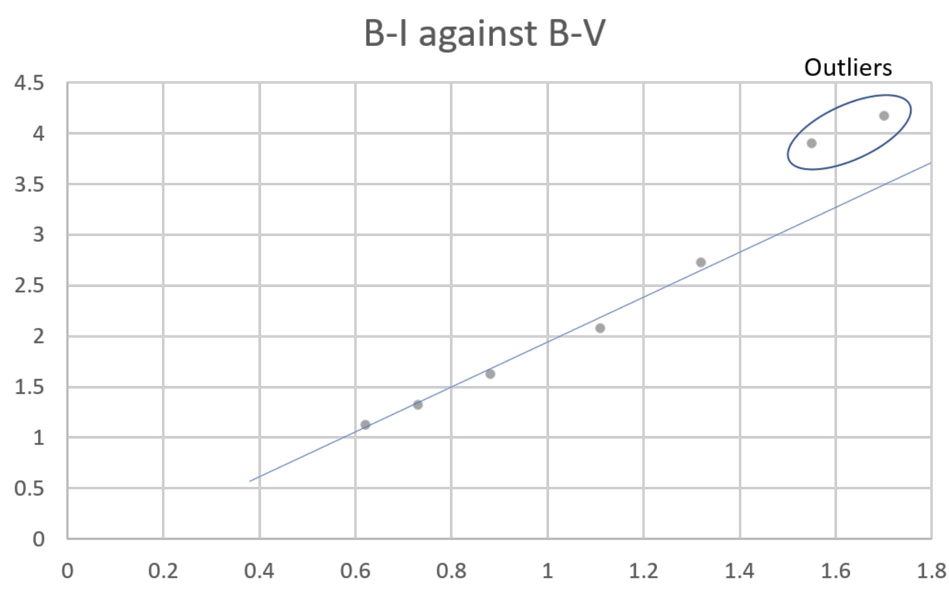
Your next task is to analyse this data. Somewhere to your side, Matou starts making a sound like a dying cat.

(b) From analyses of open clusters, it is expected that a colour-colour diagram displays a linear relationship. On a **B–I against B–V graph**, do the following.

(i) Plot the seven stars in Table 2. **[2½]**

(ii) Draw a line of best fit and identify the outlier(s). **[1½]**

Solution:



You may assume that all stars in Table 2 lie on the main sequence. In general, for main sequence stars, the **V absolute magnitude against B–I plot** is linear for B–I between 0 and 5 (the A0V and M0V spectral classes have B–I index values approximately 0 and 3.6 respectively). Your own **B–I against B–V** graph has deviations from linearity, implying that there is a certain factor that the B–V readings are more sensitive to as compared to the B–I readings.

(c) With the aid of your graph and of Table 2, as well as your own knowledge, deduce a possible candidate for this factor. Explain your reasoning. **[2]**

Solution:

Notice that AP20 and AP60 are outliers, and they are the dimmest stars of the bunch (same cluster, approximately same distances).

This makes them *cool stars*. A possible candidate factor is therefore the amount of hydrogen H_2 in the stellar atmosphere, since temperatures are cool enough for significant amounts of molecular hydrogen to form. Atmospheric composition can certainly affect colour readings.

Note: Any answer playing on the coolness of the two stars will be considered favourably. The proper explanation is indeed molecular hydrogen and makes use of mixing length (see Copeland et al. 1970), but this is of course not tested. The point of the question is to appreciate the relation of coolness with lower magnitude, and pushing that relation to something that affects luminosity.

Note that this is in fact the HR diagram's 'dip' in the main sequence towards the lower right.

Part II A Certain Scientific Rail/Gun

Your ~~ordeal~~ experience with Matou has ended. Now, Kotomi and Akuta have joined forces to, in two halves, introduce you to a study of the cosmic ray energy spectrum.

"Hello again!" Kotomi greets you cheerfully, a tag stating *Cosmic Ray Energy Lab, Level 5* dangling from her lanyard. "Since I love electromagnetism, I'll be the one introducing you to shocks and magnetic reconnection. Then Akuta will discuss the energy spectrum, okay?"

A Shocking Topic

Many astrophysical phenomena are associated with shocks. Examples include supernovae and the bow shock at the magnetosphere. A *shock*, or *shock wave*, is defined as a disturbance in a fluid moving faster than the local speed of sound, such as a plane travelling through air at supersonic speeds compressing air in front of it to form a pressure (and shock) front. When passing through a shock, pressure, temperature, and medium density are nearly discontinuous with both sides of the shock.

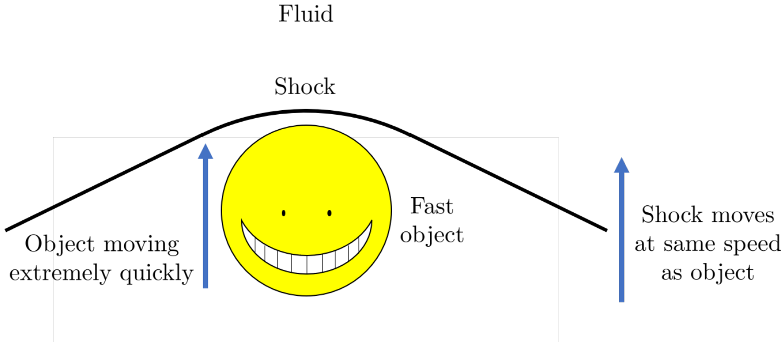


Figure 4: A fast-moving object in a fluid and associated shock.
Note that object needs to move faster than the local speed of sound to create a shock.

In media such as air or water, the primary form of energy transfer stems from collision of particles. In the low density of space, however, most shocks are *collisionless*. That is, shocks form in fluids (primarily plasma) where energy transfer between particles are mediated through means other than collisions. An example of this is the bow shock formed by the interaction of the solar wind with Earth's magnetosphere.

"Okay! A question for you!" Kotomi smiles sweetly.

(d) Briefly explain how energy transfer might occur between charged particles in low-density plasma. **[1]**

Solution:

The motion of charged particles generates a magnetic field. Another moving charged particle in such a field experiences the well-known Lorentz force. In this way energy is transferred via the field.

OR

Charged particles form an electromagnetic field permeating the plasma. As particles travel, the electromagnetic field distribution changes. Accordingly, other particles in the field will experience disturbances. In this way energy is transferred via the field.

Note: Any explanation appreciating the fact that a) particles are affected by an electromagnetic field and b) this 'affected' translates to a transfer in energy will be favourably considered. Specifics such as wave-particle resonance, Landau damping, electron cyclotron frequency etc. are NOT required.

"Here is the important picture," Kotomi says. "Because of the discontinuity, we can treat the two sides of the shock as 'walls', or 'mirrors'. A particle trapped between the two will end up bouncing between them. We can think of each collision as an elastic particle-wall collision, but with a small difference."

Figure 4 demonstrates the shock travelling through the fluid from an observer's frame of reference. It is therefore clear that from the shock's frame of reference, particles in the fluid must travel from *upstream*, i.e. in front of the shock where the particles have not yet encountered the shock, to *downstream*, i.e. behind the shock where the particles have encountered the shock.

"The mirrors' *positions* are stationary relative to the shock," Kotomi explains, holding up a finger in a lecturing pose. "But in this frame, the net velocity of the *particles* in each mirror isn't zero. There is a continuous movement of particles from upstream to the downstream, after all. What this means is that a trapped particle will bounce off mirrors that only *look* stationary."

Her hand suddenly slams into the screen, now displaying Figure 5. "But looking is not the same as being! Because the particles in the mirror have non-zero net velocity, any collision with that mirror will be like a collision with a moving wall with that net velocity! Please remember this!"

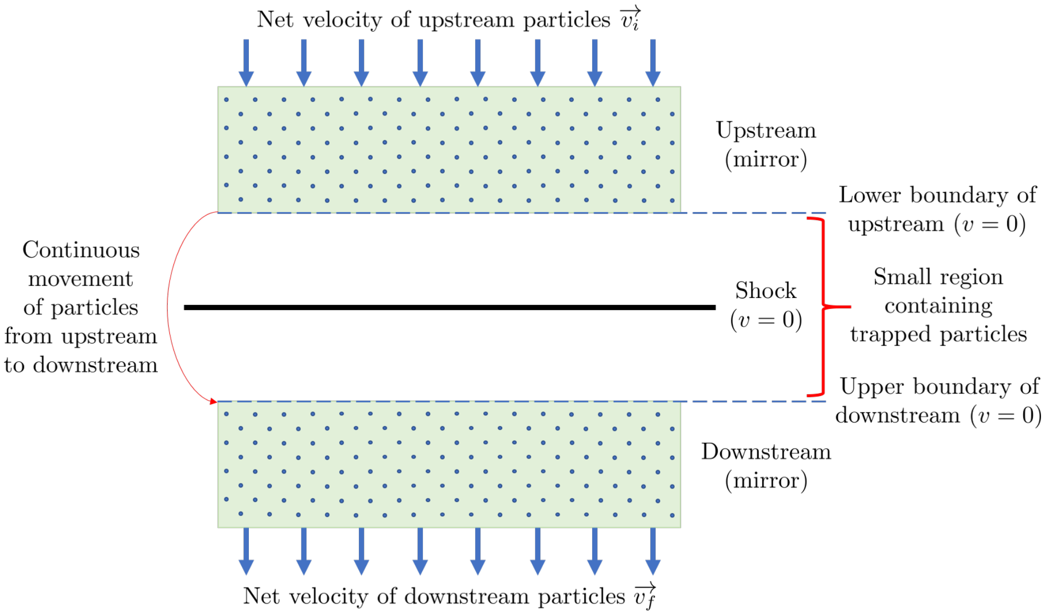


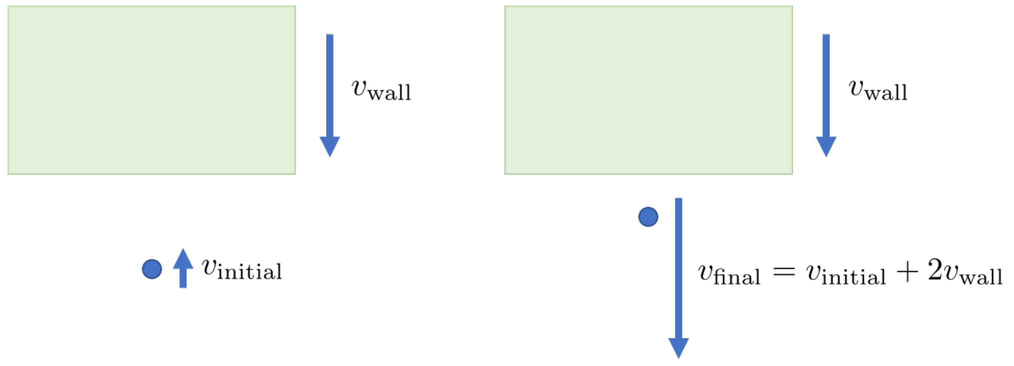
Figure 5: A high-zoom (not to scale) representative picture of the mirrors and the region around the shock. Velocities are taken in the reference frame of the shock.

Recall that for an elastic collision, relative speed is always conserved.

(e) Assume that both $\vec{v_i}$ and $\vec{v_f}$ are directed downwards. It is standard for shocks that $v_i > v_f$. Consider a non-relativistic particle P in the reference frame of Figure 5 with initial speed V oriented upwards. Explain why, in the absence of other forces, P will continually bounce between both mirrors, and that the speed of P per cycle (i.e. per two collisions, one with each mirror) increases by $2(v_i - v_f)$. You may wish to use diagram(s) to aid your explanation. **[2]**

Solution:

We consider the solution in terms of moving walls.



The upstream mirror is a massive wall with downward speed v_i, and P has initial velocity V upwards. Hence the relative speed is $V + v_i$. Since relative speed is conserved and the velocity of the wall does not change, the final velocity of p is $V + 2v_i$ oriented downwards after elastic collision.

Since $v_i > v_f$, P is moving downwards *faster* than the downstream wall and thus will rebound off the wall. The relative speed of P with this wall is $V + 2v_i - v_f$. Therefore the final speed of P after collision is $V + 2v_i - 2v_f$ in the upwards direction.

Since $v_i > v_f$, the particle is indeed travelling upwards with speed $> V$. This sets up a bouncing, and the speed gain is $2(v_i - v_f)$ per cycle.

Note: One might argue that the particle will always rebound off the downstream mirror, regardless of how we consider the wall picture. It is true that the particle must always collide with the downstream mirror. However, whether it *rebounds* is a different story. If it were the case that $v_{\text{wall}} \geq v_{\text{particle}}$, then no energy transfer can take place to make the particle rebound. The case of $v_{\text{net}} \geq v_{\text{particle}}$ in the downward case corresponds to this exact case, and no energy transfer to direct the particle upwards takes place.

Kotomi's Magnetic Personality

"We're done with shocks!" Kotomi says happily. "Now, let's talk about magnetic reconnection!"

Magnetic reconnection is the process whereby separate magnetic field lines join together and cause a change in the distribution (or topology, if one wants to be pedantic) of magnetic field lines. This process is exceedingly common in plasmas in space, where magnetic field lines can switch directions often.

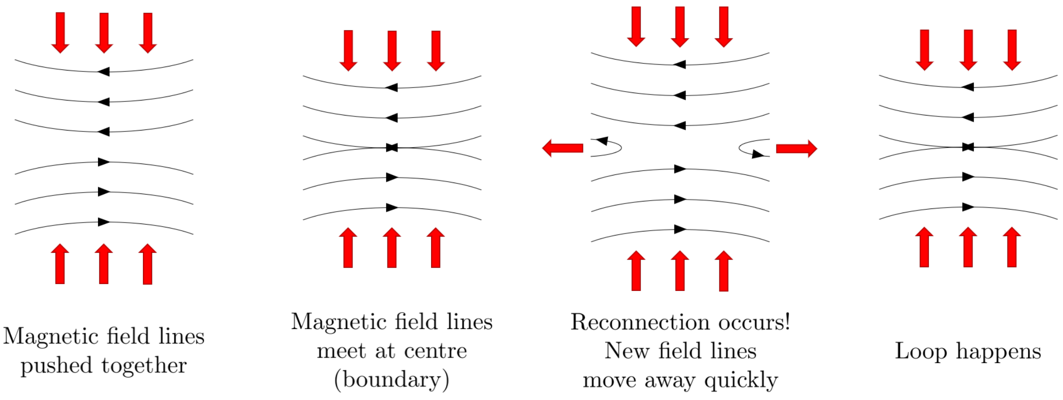


Figure 6: Magnetic reconnection process in a nutshell.
Inline arrows dictate magnetic field lines' directions.
Thick arrows represent direction of motion of magnetic field lines.

Generally speaking, there is a region of space where oppositely-directed magnetic field lines are pushed together and meet. This causes the field lines to merge, splitting off to form two new field lines. The new lines 'rebound' in perpendicular directions, much like the snapping of a rubber band.

This has two effects.

1. It creates a 'magnetic vacuum' at the boundary. The original sets of oppositely-directed magnetic field lines are drawn in by the magnetic tension to continue merging at the boundary, forming a looping process.
2. It causes charged particles to accelerate to potentially high velocities, while imparting a significant amount of thermal and kinetic energy to said particles.

The second point is particularly relevant, as the resultant effect is a burst of fast-moving charged particles. This makes magnetic reconnection an important component of various phenomena such as solar flares and geomagnetic storms.

"The acceleration that forms cosmic rays is caused by charged particles repeatedly crossing the boundary," Kotomi lectures. "In a sense, the particles *bounce* between both sides."

(f) Briefly explain TWO possible reasons why a charged particle might repeatedly cross the boundary. **[2]**

Solution:

Reason 1: Charged particles tend to spiral around magnetic field lines. This creates a scenario in which if the field lines are sufficiently close to the boundary, the spiralling allows the particles to cross the boundary.

Reason 2: The high kinetic motion may result in the event that the particle does not have a tight helical spiral. In this case it is possible that the particle, if oriented nearly normal to the boundary, has enough energy to cross the boundary and repeatedly collide with alternating walls and reflected off each.

Reason 3: A similar reason as Reason 2 but arguing on high kinetic random motion instead.

Note: Any two suitable reasons will suffice.

Part III A Certain Shocking Accelerator

Akuta then shows up with a sneer. “Feh. Name’s Akuta. Pseudonym. I do stuff with vectors. Oi, Kotomi. Ya done with the easy stuff? Now I gotta explain the Fermi rubbish?”

“It’s Fermi *acceleration*,” Kotomi bites back with a slight frown. “Please treat this seriously, Akuta-san.”

“Tch. Whatever.”

“Akuta-san!” Kotomi moves to grab his ear. “Do! This! Properly!”

“Ow! Okay, okay! Fine already! Ow! Stop it, you barbarian!”

“Start with cosmic rays!” Kotomi insists sternly. “Got it?”

“I got it already!! Get off! Ow!”

Space Accelerator Fermi

Cosmic rays are high-energy particles moving through space at high relativistic speeds. They primarily consist of nuclei, with a small proportion being electrons and a tiny amount being other particles.

(g) State two astronomical sources of cosmic rays. **[1]**

> **Solution:**
>
> Supernovae, CMEs, radio galaxies, colliding galaxies, solar wind, etc.
>
> **Note:** It’s a free question. Any two sources will suffice.

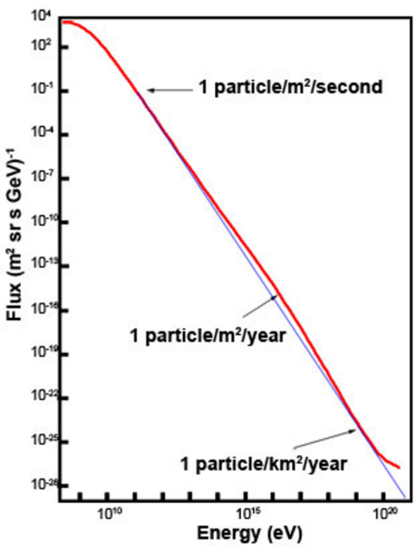


Figure 7: Energy spectrum of cosmic rays.
Thick line: Observed spectrum.
Thin line: Exponential decay line.

The energy distribution, or spectrum, of cosmic rays is a subject of much interest. In general terms, the flux of cosmic rays decreases rapidly as energy of the cosmic rays increases. That is, cosmic rays at lower energies are much more common than cosmic rays at higher energies. The decline (Figure 7)[6] follows a power law with index approximately −2.8.

“The main explanation for this exponential decay is Fermi acceleration,” Akuta explains sourly while rubbing his ear. “It’s the acceleration of charged particles under multiple reflections.”

The original idea of cosmic ray generation, proposed by Fermi, is the reflection of particles off ‘magnetic mirrors’ multiple times. This concept is, of course, closely linked to the concept of mirrors in shocks. We have seen (cf. Figure 5) that a particle bouncing off mirrors repeatedly can be accelerated.

Magnetic reconnection sites are of particular interest in this case. Cosmic rays can be generated at such sites by a ‘basic’ version of Fermi acceleration. Such sites are also relatively easy to study using the concepts of shocks, and under shock assumptions the acceleration produces an energy spectrum with index −2.5. It is this simple picture that we will study.

“Simple’s good,” Akuta grunts with clear irritation at the idea of an overly-complicated picture. “Yeah. Simple’s good.”

[6] Modified from https://astronomy.swin.edu.au/cosmos/c/cosmic+ray+energies.

It's All About Controlling Vectors

"Here's how we see magnetic reconnection as a shock." Akuta brings up Figure 8.

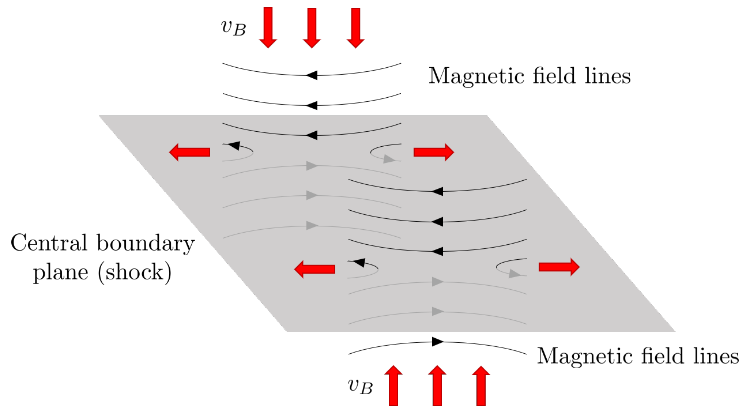


Figure 8: 3D view of magnetic reconnection with planar central shock boundary (cf. Figures 5 and 6). Velocities are taken in the reference frame of the central boundary plane.

In 3D, the central boundary plane at which magnetic field lines meet can be taken to be a stationary shock (cf. Figure 5). The magnetic field lines move, hence the charged particles carried by the field lines move as well. As a consequence, the field lines above and below the central boundary plane are 'magnetic mirrors', and can be taken to be mirrors similar to those introduced in Figure 5. We may assume that the field lines (and hence the particles in the mirrors) are moving towards the central boundary plane non-relativistically at speed v_B each.

(h) A particle P with speed v very close to c in the reference frame of the central boundary plane is approaching the mirror at angle θ to the normal, with initial energy E and magnitude of momentum p. P then collides with the mirror elastically. Let E'' be the final energy of P in the reference frame of the central boundary plane.

(i) **(BONUS)** Show that after the collision, in the frame of the shock, P has final energy

$$E'' = \gamma_B(E' + v_B p'_x),$$

where γ_B is the Lorentz factor with respect to speed v_B, $E' = \gamma_B(E + v_B p\cos\theta)$, and $p'_x = \gamma_B\left(p\cos\theta + \frac{v_B E}{c^2}\right)$. **[2]**

Solution:

Recall the (vector) equations for the energy-momentum Lorentz transform

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{v\mathbf{n}\cdot\mathbf{p}}{c}\right),$$

$$\mathbf{p}' = \mathbf{p} + (\gamma - 1)(\mathbf{p}\cdot\mathbf{n})\mathbf{n} - \frac{\gamma E v\mathbf{n}}{c^2},$$

where $\mathbf{n}$ is the direction of relative motion in the transformation.

By direct substitution, the Lorentz transform of energy from the shock frame to the centre of momentum (COM) frame is $E' = \gamma_B(E + v_B p\cos\theta)$, with $\mathbf{n}$ negative. Also by direct substitution in the momentum Lorentz transform equation, the x-component of momentum in the COM frame (i.e. normal to the mirror) is $p'_x = \gamma_B\left(p\cos\theta + \frac{v_B E}{c^2}\right)$.

In the COM frame, the collision results in the momentum in the normal direction being reversed while particle energy is conserved, i.e. $E' \mapsto E'$ and $p'_x \mapsto -p'_x$. Hence by transforming back to the shock frame, the final particle energy in the shock frame is $E'' = \gamma_B(E' + v_B p'_x)$.

(ii) Show that the fractional energy gain of P per collision is approximately

$$\frac{\Delta E}{E} \approx \frac{2v_B \cos\theta}{c}.$$

(Hint: Recall the relativistic energy and momentum relations $E = \gamma mc^2$ and $p = \gamma mv$.) **[3]**

Solution:

Performing a substitution from the above, the final energy in terms of the initial quantities is

$$\begin{aligned} E'' &= \gamma_B(E' + v_B p'_x) = \gamma_B\left(\gamma_B(E + v_B p\cos\theta) + v_B\gamma_B\left(p\cos\theta + \frac{v_B E}{c^2}\right)\right) \\ &= \gamma_B^2\left(E + v_B p\cos\theta + v_B p\cos\theta + \frac{v_B^2 E}{c^2}\right) \\ &= \gamma_B^2 E\left(1 + \frac{2v_B p\cos\theta}{E} + \frac{v_B^2}{c^2}\right). \end{aligned}$$

The energy change is

$$\Delta E = E'' - E = E\left(\gamma_B^2\left(1 + \frac{2v_B p\cos\theta}{E} + \frac{v_B^2}{c^2}\right) - 1\right).$$

Applying $v_B \ll c$, we have $\frac{v_B^2}{c^2} \approx 0$ and therefore $\gamma_B \approx 1$, so that

$$\frac{\Delta E}{E} \approx \frac{2v_B p\cos\theta}{E}.$$

The final step is using the provided relations to obtain the formula $v = \frac{pc^2}{E}$. With this substitution we arrive at

$$\frac{\Delta E}{E} \approx \frac{2v_B v\cos\theta}{c^2} \approx \frac{2v_B\cos\theta}{c},$$

using $v \approx c$.

"So far ya've seen a single particle do stuff," Akuta continues. "But there's more to it. When ya have many particles, they don't all move the same way."

For a large number of particles in three dimensions, it is reasonable to assume that the motion of particles are distributed isotropically. For an isotropic distribution, the probability density[7] $p(\theta)$ of finding a particle at an angle θ to the normal of the central boundary plane satisfies $p(\theta) = 2\sin\theta\cos\theta$.

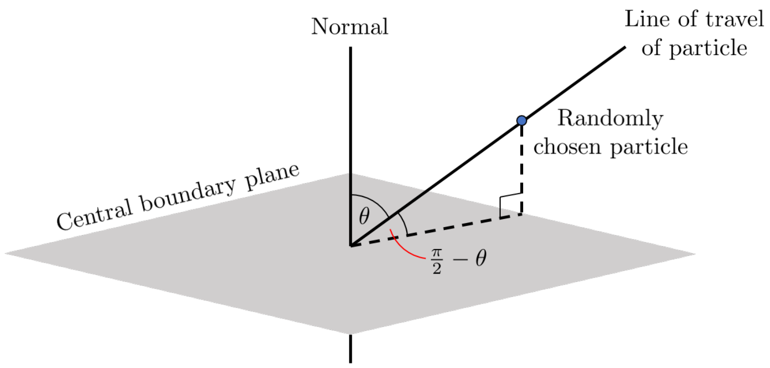


Figure 9: Diagrammatic representation of the angle θ of a particle with respect to the normal. In an isotropic distribution, for any fixed $0 \le \theta_0 \le \frac{\pi}{2}$, the probability density at $\theta = \theta_0$ is $p(\theta_0) = 2\sin\theta_0\cos\theta_0$. Note that since the lines in consideration are undirected, therefore $0 \le \theta \le \frac{\pi}{2}$ always.

(j) For a large number of particles satisfying the speed condition of Part (h), show that the average fractional energy gain per particle per collision is approximately

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4v_B}{3c}.$$

[1]

Solution:

Average energy gain per crossing is found by integrating over the probability:

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \int_0^{\frac{\pi}{2}} \frac{2v_B\cos\theta}{c} \cdot 2\sin\theta\cos\theta\, d\theta = \frac{4v_B}{c}\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\, d\theta = \frac{4v_B}{c}\left[-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} = \frac{4v_B}{3c}.$$

Note: One might raise the argument that we should only consider particles heading towards the mirror. This is fine. The plane of the mirrors are taken to be parallel to the central boundary plane, so we only need to consider particles heading towards the central boundary plane. An isotropic distribution implies precisely half the particles are heading towards the central boundary plane. But since both the total number of particles and the density of particles on any given line of travel is halved, there is no overall change to $p(\theta)$.

"Now for the exit." Akuta grins unpleasantly. "They exit at different times. So they don't all got the same final energy."

Ultimately, particles do not stay bouncing between the two mirrors indefinitely. Due to the flux or particle flow, particles must eventually exit the shock region – this is true for all shocks. In the case of magnetic reconnection, charged particles are swept to the side of and away from the boundary by the magnetic field lines moving away from the reconnection site.

(k) Under shock assumptions, the fraction of particles lost per collision is approximately $\frac{2v_B}{c}$. Assume that particle loss is independent of particle energy. Let $N(E)$ be the number of particles accelerated to an energy at least E. Show that

$$\frac{dN}{dE} \approx kE^{-2.5},$$

[7] That is, the probability of finding the particle at an angle between θ and $\theta + d\theta$ is $p(\theta)d\theta$.

where k is a constant. **[4]**

Solution:

Let $\beta = 1 + \frac{4v_B}{3c}$. Then after n collisions, a particle with initial energy E_0 is expected to have energy $E = E_0\beta^n$. Since $\frac{2v_B}{c}$ of particles are lost after each collision, set $p = 1 - \frac{2v_B}{c}$, and after n collisions we expect $N = N_0 p^n$ particles to remain.

Since loss is independent of energy, we have

$$\frac{\log N - \log N_0}{\log E - \log E_0} = \frac{\log \frac{N}{N_0}}{\log \frac{E}{E_0}} = \frac{n \log p}{n \log \beta} = \frac{\log p}{\log \beta}.$$

Rearranging,

$$N = \frac{N_0}{E_0^{\frac{\log p}{\log \beta}}} E^{\frac{\log p}{\log \beta}} = k_0 E^{\frac{\log p}{\log \beta}},$$

where k_0 is a constant.

This is the relation for number of particles expected to eventually have energy E or greater. So

$$\frac{dN}{dE} = \frac{\log p}{\log \beta} k_0 E^{\frac{\log p}{\log \beta}-1} = kE^{\frac{\log p}{\log \beta}-1},$$

where we have written the constant $k = \frac{\log p}{\log \beta} k_0$.

Finally, using first order approximation,

$$\frac{\log p}{\log \beta} \approx \frac{-\frac{2v_B}{c}}{\frac{4v_B}{3c}} = -\frac{3}{2},$$

so that

$$\frac{dN}{dE} = kE^{-2.5}.$$

“And this,” Akuta concludes, “is a power law with index −2.5. Course, the actual cosmic ray picture’s a fair bit more complicated.” He snorts at the understatement. “But eh. Now you know the basics.”

Epilogue: A Certain Ending Testament

This part serves as the story ending ONLY. There are NO MARKS in this part.

You come away from the tour, your head wildly spinning. You certainly feel the drain on your body! Has your knowledge been thoroughly tested?

Has your brain been thoroughly blended?

As you, Kotomi, and Akuta arrive at the designated ending point of your tour, you see Matou scratching his head sheepishly. Kotomi moves to his side, Akuta following a second later with a small "tch".

"Umm...so...thank you for coming today," Matou says, fidgeting somewhat awkwardly. "We hope you've enjoyed your time today in our city, and we hope you'll consider coming here for your future studies or to research. We hope we've been adequate hosts today and if you have any complaints, please feel free to bring them up with–"

As he continues speaking, you notice Kotomi getting steadily more exasperated.

"Too long, idiot!" Kotomi smacks his head before she turns to you while re-adopting a flowery smile. "I humbly apologise. We thank you and we hope your day has been fruitful. We hope to see you again sometime!"

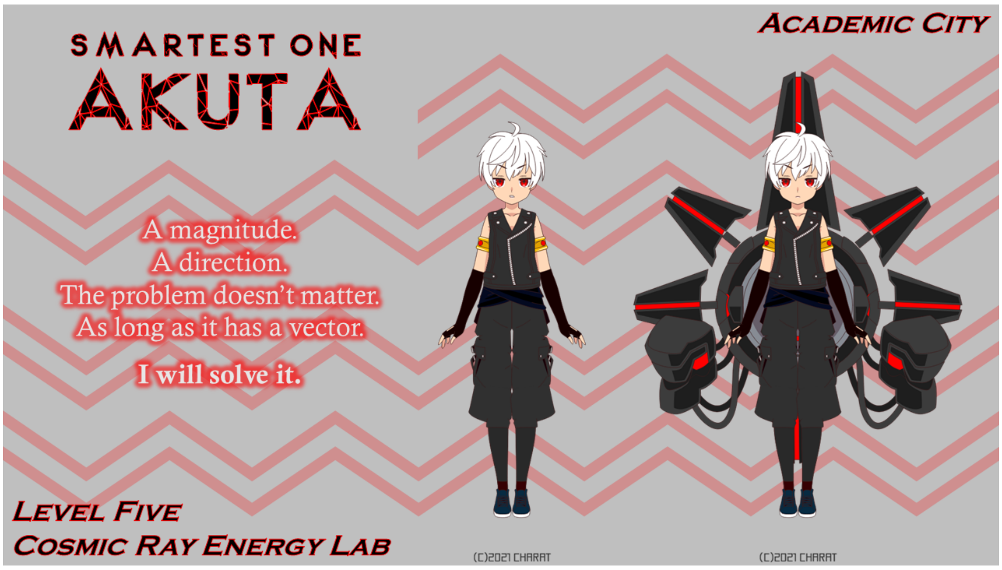
Then she smirks conspiratorially and adopts a stage whisper. "And if you have any *feedback* about *these two guys...*"

"Ya lookin' for a fight!?" Akuta snarls.

It isn't long before the three are bickering once again.

∼ FIN ∼

Bonus: Character Cards!



Character art due to CHARAT. Card designs due to the question setter.

[End of paper]