AstroChallenge 2024
Senior Team Round

SOLUTIONS

Monday 3rd June 2024

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper has a total of **44** printed pages, including any blank pages and this cover page.
2. Any materials other than the Question Paper, Formula Booklet, and **ONE** A4-sized cheat sheet per team, are strictly prohibited.
3. Do **NOT** turn over this page until instructed to do so.
4. You have **2 hours** to attempt **ALL** questions in this paper.
5. Write your answers on blank pieces of A4 paper or graph paper (if necessary).
6. Use a separate piece of paper for each question; no one piece of paper should contain solutions to more than one question.
7. The marks for each question are given in brackets in the right margin, like such: **[2]**.
8. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).
9. At the end of the paper, submit your answer script with solutions ordered accordingly. You do not need to submit this booklet.
10. Ensure that your school and team number are clearly indicated in your answer script.
11. It is **your team's** responsibility to ensure that all pages of your answer script have been submitted, including pages to be detached from this booklet.

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Question 1 Mercury Exploration [20 marks]

Part I Debunking Myths with Science

(a) Consider the following statement:

"The duration between successive retrograde motions of Mercury is __?__ the orbital period of Mercury which is 88 days."

(i) Fill in the blank above with one of these options: shorter than / equals to / longer than. [1]

Solution:

longer than

(ii) Explain your answer for (i). [2]

Solution:

Retrograde motion of an inferior planet occurs when it is in inferior conjunction, overtaking the Earth [1]. Thus, the time between successive conjunctions would be the orbital period of Mercury (88 days) and the additional time required to catch up with the distance Earth moved in those 88 days [1]. This actual synodic period if 116 days.

Part II The Challenges of Travelling to Mercury

(b) Mercury's orbital speed varies from 39 km/s to 59 km/s. Earth has an orbital speed of 30 km/s and does not vary much.

Explain why Mercury's orbital speed is higher than that of Earth's, and why its orbital speed varies throughout its orbit. [2]

Solution:

As Mercury is very close to the sun, high velocity is required to maintain its orbit [1]. Furthermore, Mercury has a highly eccentric orbit and thus the orbital speed is higher near the perihelion while lower near its aphelion [1].

(c) **(i)** Briefly describe where the Lagrange points of a celestial body orbiting a parent star are. [1]

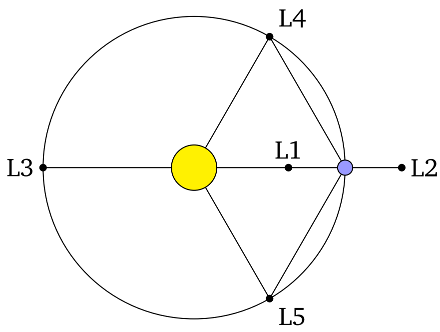
Note. *You may opt to provide a technically accurate illustration for your answer.*

Solution:

Option 1: Description

The Lagrange Points in the Mercury-Sun system are located at five specific positions where the gravitational forces and centrifugal force balance each other. L1, L2, and L3 are collinear with the Sun and Mercury, while L4 and L5 form equilateral triangles with Mercury and Sun.

Option 2: Drawing



(ii) List two potential benefits an orbit around a parent star at one of the Lagrange points will have compared to geosynchronous orbits around a celestial body orbiting a parent star. [2]

Solution:

1. Continuous observation of the parent star possible, unlike geosynchronous orbits where the satellite will go behind Earth. For example, SOHO at L1.
2. Continuous coverage from parent star for deep sky observation. These telescopes are very sensitive to infrared; thus, geosynchronous orbits will render them useless for some time.

[Any other physically correct reason would also secure points]

Launched in 1973, Mariner 10 was the first spacecraft to conduct a Mercury fly-by at a closest approach of 704 km. Years later in 2004, the MESSENGER spacecraft orbited Mercury collecting critical data, providing insights to the Mercurial surface and magnetosphere.

(d) (i) Assess how Mercury's orbital eccentricity and proximity to the Sun would affect a spacecraft sent from Earth to Mercury. [1]

Solution:

Mercury's elliptical orbit and close proximity to the Sun result in stronger and more variable gravitational forces, hence requiring precise trajectory calculations.

[Any other physically correct reason is also accepted]

(ii) Briefly describe how the challenges experienced by a spacecraft sent to Mercury would differ from one sent to Mars. [2]

Solution:

Mars' orbit is much less eccentric and it is further away from the Sun, thus the gravitational force variability is low. This necessitates precise trajectory planning and greater fuel requirements for navigation and corrections for Mercury, but not for Mars.

[Any other physically correct reason is also accepted]

C3 energy represents the kinetic energy a spacecraft needs to escape the gravitational influence of a celestial body (e.g. Earth) and enter a heliocentric orbit. A Hohmann transfer orbit is an efficient method for moving a spacecraft between two orbits around the same primary body, involving two engine burns to transition from a lower orbit to a higher one through an elliptical transfer orbit. Bi-elliptic transfers are useful for very large changes in orbital altitude involving three burns and two elliptical orbits.

(e) **(i)** What is Mercury's escape velocity, and how does it compare to that of Earth's? [1]

Solution:

Mercury's escape velocity is about 4.25 kilometers per second, which is lower than Earth's 11.2 kilometers per second.

Calculation is not required:

At escape velocity, an object will completely leave the gravitational field of a system, therefore,

$$E_{total} = 0 \Rightarrow E_{KE} = -E_{GPE} \Rightarrow \frac{1}{2}mv_{esc}^2 = \frac{GMm}{R}$$

where M is the mass of the planet, m is the mass of the object, and R is the radius of the planet. Therefore,

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

(ii) Briefly describe how the differing escape velocities of Mercury and Earth influence the energy requirements for spacecrafts attempting to leave their surfaces and achieve orbital transfers to other planets. [2]

Solution:

The greater the escape velocity from Earth implies it takes a larger amount of fuel for a body to be accelerated to the escape velocity. Thus, less fuel is required when escaping Mercury as compared to Earth.

[Any other physically correct reason is also accepted]

(f) Describe the effects of solar radiation pressure on a spacecraft's trajectory and mission planning near Mercury? [1]

Solution:

Solar radiation pressure exerts a significant force on a spacecraft near Mercury due to its close proximity to the Sun. This force must be accounted for in trajectory calculations and mission planning to ensure accurate navigation.

[Any other physically correct reason is also accepted]

(g) Briefly describe potential challenges faced by a spacecraft landing on Mercury compared to landing on other planets with thicker atmospheres (e.g., Mars or Venus)? [1]

Solution:

Mercury's thin atmosphere offers no aerodynamic deceleration/atmospheric braking, requiring precise propulsion manoeuvres for landing.

(h) What thermal conditions would a spacecraft encounter near Mercury, and how would these temperatures influence spacecraft design and material choice? [2]

Solution:

Near Mercury, a spacecraft would encounter extreme temperatures ranging from about 430°C on the sunlit side to −180°C on the dark side. These conditions necessitate advanced thermal protection systems and materials capable of withstanding drastic temperature variations.

[Any other physically correct reason is also accepted]

(j) **(i)** What are the effects of Mercury's magnetic field on the operation and data collection of an spacecraft orbiting Mercury. [1]

Solution:

Mercury's weak magnetic field can still affect the operations and data collection of an orbiting probe by causing fluctuations in the local magnetic environment.

[Any other physically correct reason is also accepted]

(ii) Suggest a mitigation measure to reduce these effects. Briefly explain your suggestion. [1]

Solution:

Instruments must be shielded and calibrated to account for these variations to ensure accurate data collection.

[Any other physically correct measure is also accepted]

Question 2 Galactic Cartography [20 marks]

Part I What's so difficult about it? [1 mark]

According to our best models of our Milky Way, it seems to be a barred spiral galaxy. Unlike other galaxies far out in space, we cannot directly observe the shape and structure of our own galaxy. However, we can at least observe some basic facts about its shape. For instance, the Milky Way appears as a distinct and dusty strip crossing the night sky, and thus it must be a flat disk with the Solar System located within the disk plane. The Milky Way must also be a spiral galaxy since it has a large proportion of young star clusters relative to older elliptical galaxies.

As for the detailed structure of the Milky Way, we can only deduce it through various sorts of indirect observations. For example, astronomers have measured the locations and distances of young star clusters to determine the spiral arm structure of the Milky Way. Using this method, astronomers have been able to map the structure of the nearby arms such as the Orion and Perseus arms (Figure 1). However, this method is not suited for mapping structures further away from the Solar System.

(a) Suggest a reason why the star cluster method is only suitable for mapping the region near the Solar System, and not any further. [1]

Solution:

The disk of the Milky Way is filled with interstellar dust. We are unable to observe star clusters too far away from the solar system as they are blocked by the interstellar dust [1].

OR

At further distances, extinction due to dust is significant and difficult to model, leading to large uncertainties in the distance determination of star clusters [1].

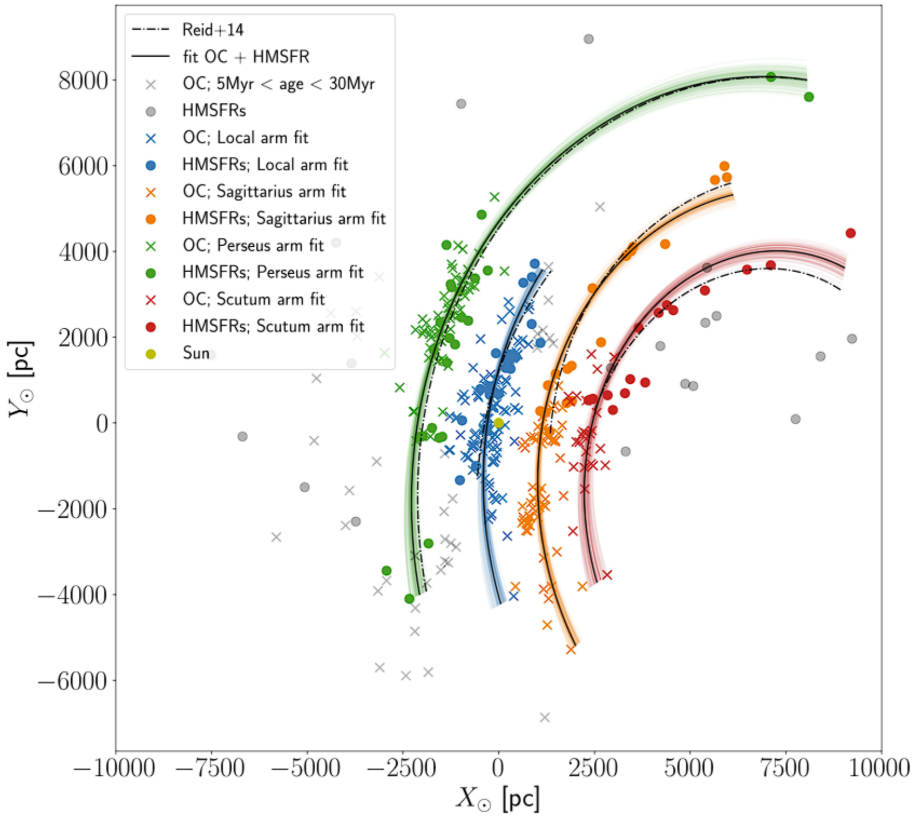


Figure 1: Spatial distribution of young open star clusters from a recent Gaia data release. Note that the data points are restricted to around the local Orion arm.

In 1958, Danish astronomer Jan Oort published a map of the Milky Way based on his observations of hydrogen emissions[1]. The map (Figure 2) is a contour plot of neutral hydrogen densities, with darker regions representing higher density. The cross in the centre marks the location of the galactic centre, and the concentric circles mark lines of constant distance from the galactic centre. The Solar System is located top of centre and marked by an empty circle. A full-page version of the map can be found in the appendix on Page 15.

Remarkably, the map covers a large proportion of the galactic disk as viewed top-down from the north galactic pole. We can also identify long density filaments corresponding to spiral arm structures in the Milky Way.

[1] Oort, J. H., Kerr, F. J., & Westerhout, G., 1958, MNRAS, 118, 379, doi:10.1093/mnras/118.4.379

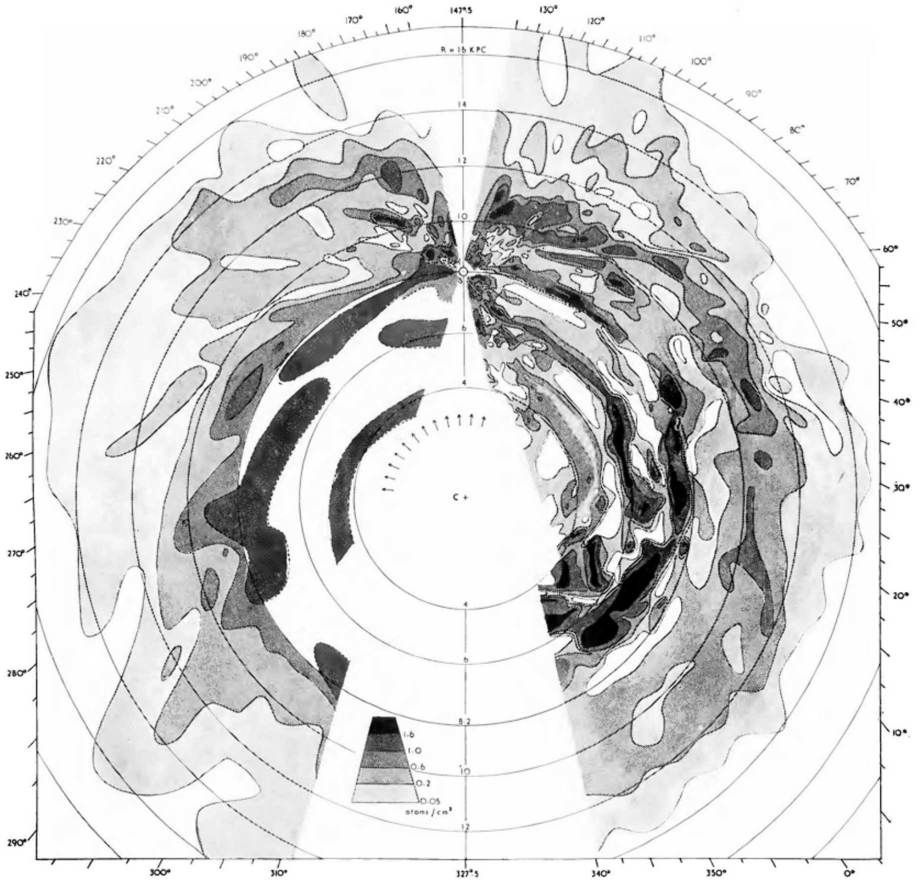


Figure 2: Oort's (1958) map of the distribution of neutral hydrogen in the Milky Way. The view is from the galactic north pole, and the edge degree markings mark out the galactic longitude. Note that the longitude system used here was an older system that defined the 0° longitude to be at the intersection between the equatorial and galactic planes, before its redefinition by the IAU in 1958 as the direction of the galactic centre.

So how did Oort manage to construct his map of the Milky Way? What were the observational data that he used, and how did he transform them into a density plot in space? This is what we are going to explore in this question.

Part II The Astronomer's Toolkit [4 marks]

The spectrum of a light source tells us the intensity distribution of the source over different wavelengths or frequencies of light. As such, spectroscopy, or the study of light spectra, is an essential tool of astronomers. The spectra of an astronomical source can yield information on the chemical composition of the source. But we can also flip things around. By restricting observations to a emission wavelength characteristic of a known substance, we can thereby infer the distribution of this specific substance across a large area of sky.

This is what Jan Oort did, and he specifically looked in the radio wavelengths for the 21-cm line emission associated with the spin-flip transition in neutral atomic hydrogen. A more intense 21-cm line emission is, broadly speaking, related to a higher gas density along the line of sight. The long radio wavelength of the 21-cm line allows astronomers to map out the hydrogen gas structure of the Milky Way to a greater extent than what had been possible with the star cluster method.

(b) How does the long wavelength of the 21-cm emission line allow us to map a larger portion of the Milky Way than the star cluster method? [1]

Solution:

Interstellar gas and dust in the galactic plane tend to scatter light in the optical wavelengths, but is transparent to radiation in longer infrared and radio wavelengths [0.5]. Observing in radio wavelengths thus allows us to 'see' through the dust clouds and observe the structures on the far side of the Milky Way [0.5].

The 21-cm line occurs because the lone valence electron in hydrogen can occupy two different spin states with very slightly different energies. If the electron is originally in a higher-energy spin state, there is a very small chance over a long period of time for the electron to de-excite, releasing a photon with a 21cm wavelength. Neutral hydrogen has an excitation lifetime of around 1.1×10^7 years. Due to the energy-lifetime uncertainty principle, the width of the 21-cm spectral line is extremely narrow. This makes the 21-cm line very suitable for Doppler spectroscopy, which makes use of the redshift or blueshift of radiation from a source to determine its line-of-sight (radial) velocity.

(c) Why is the narrow spectral width of the 21-cm line advantageous for Doppler spectroscopy? [1]

Solution:

The narrow width of the spectral line means that we can distinguish very small redshift or blueshift magnitudes in emission frequency/wavelengths [0.5], therefore allowing for sensitive and accurate velocity measurements [0.5].

Doppler spectroscopy has been commonly used to map the rotation curve of galaxies by observing the spectroscopic redshift or blueshift along the diameter of a galaxy. Jan Oort found a different use for Doppler spectroscopy; rather than measuring the overall velocity curve of the Milky Way, he used it to determine the distances to neutral hydrogen clouds along a line-of-sight.

(d) Why can Doppler spectroscopy be used to determine distances of hydrogen clouds along a line-of-sight in the galactic plane? [2]

Hint. *If you are stuck, skip ahead and come back to this question later.*

Solution:

At different distances along a single line-of-sight, the gas clouds will be at a different distance from the galactic center. As a result, they will be orbiting the galactic centre with different tangential velocities [1]. This results in distinct radial velocities that contribute to different amounts of redshift/blueshift of the hydrogen line [1].

Part III 'Fun' With Galactic Geometry [4 marks]

Imagine if you were an astronomer like Oort, trying to figure out what kind of data or observations you would need to construct a map of the Milky Way. The first step to tackling the problem would be to have a physical model of the Milky Way in mind. Since the Milky Way is a flat disk, it is convenient for us to imagine all of the stars, gas, and dust as living in a flat 2-dimensional plane. In addition, spiral galaxies rotate about their centres. For this situation, polar coordinates on the 2-D galactic plane are an obvious choice.

Let (θ, R) be the set of polar coordinates centred about the Solar System, with θ representing the longitudinal angle of the object relative to the galactic centre, and R representing the radial distance of the object from the Solar System. Let (ϕ, r) be the set of polar coordinates centred about the galactic centre, with ϕ representing the angle of the object relative to the Solar System, and r representing the orbital radii of the object around the galactic centre. The respective coordinates can be summarized in the following diagram (Figure 3), with r_0 representing the distance of the Solar System S to the galactic centre G, and P representing the point for which the coordinates are defined.

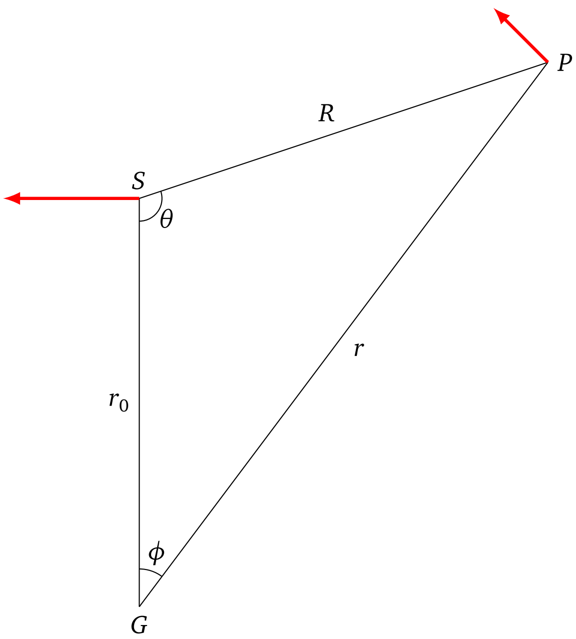


Figure 3: Diagram summarizing the geometry of the coordinates chosen. The red arrows represent the directions that the Solar System and matter at P orbit about the galactic centre in an inertial frame.

Ultimately, we need an expression for the line-of-sight radial velocity V_r relative to the Solar System as a function of some spatial coordinates. Ideally, this would be R and θ, since those coordinates are centered at the Solar System. However, a function of r and θ is actually more convenient. We can *approximately* describe the orbital motion of matter in the galactic disk with an unknown *angular* velocity rotation curve $\omega(r)$. This is only a function of the radial distance from the galactic centre. Our more familiar tangential velocity V_r is thus given by $r \times \omega(r)$. Note that since the Solar System is also in orbit around the galactic centre, the angular velocity of the Solar System $\omega(r_0)$ needs to be subtracted to obtain the relative angular velocities in the Solar System frame.

(e) Show that the radial velocity V_r can be written as the following expression: [2]

$$V_r = r_0 \sin\theta \left[\omega(r) - \omega(r_0)\right].$$

Solution:

The velocity v is given by $r \times (\omega(r) - \omega(r_0))$ [1].

We want to find v_r by projecting v onto the radial line-of-sight axis. Let us label the angle SPG α. This is our angle of interest since $v_r = v\sin(\alpha)$. By the sine rule:

$$\frac{\sin(\theta)}{r} = \frac{\sin(\alpha)}{r_0}$$
$$\sin(\alpha) = \frac{r_0}{r}\sin(\theta)$$

Substituting this into our expression for v_r, we obtain [1]:

$$\begin{aligned} v_r &= \frac{r_0}{r}\sin(\theta)v \\ &= \frac{r_0}{r}\sin(\theta)\times r\times(\omega(r)-\omega(r_0)) \\ &= r_0\sin(\theta)(\omega(r)-\omega(r_0)) \end{aligned}$$

In deriving this equation, we accounted for the motion of the Solar System around the galactic centre by subtracting a $\omega(r_0)$ term. However, this does not actually fully account for the effects of the Solar System's own motion, and Oort's observational team had to include an additional dipole correction term[2] to the observed Doppler shifts.

(f) Explain what does the $\omega(r_0)$ term actually represent, and why does subtracting it not fully account for the effects of the solar system's motion on the doppler shift observations. [1]

Solution:

On average, matter in the local neighbourhood of the solar system orbits the galactic centre at approximately $\omega(r_0)$, which provides a reference frame called the local standard of rest.

However, the solar system has an additional velocity relative to this local standard of rest [1].

This results in an additional observed Doppler shift, with maximum blueshift observed in the direction of the solar system's peculiar velocity, and redshift in the anti-pole direction.

What is really unfortunate is that r and θ do not unambiguously denote the position of a point on the galactic plane when r is smaller than the orbit of the Solar System around the galactic centre. This is because ϕ can now be either acute or obtuse, leading to two different intersection points that contribute to the same V_r. This ambiguity is inherent to the geometry of the system, so no, this cannot be resolved by switching to a different set of coordinate variables.

(g) Explain clearly why does this ambiguity lead to difficulties in mapping out the spatial density distribution along the line of sight, [1]

Solution:

For a given V_r, we are unable to determine the relative contributions in emission power from the two different points, therefore we can only determine the total combined density, but not the individual

[2] In this context, a dipole correction term means a positive correction to redshift in one direction in the sky (called the pole), and a negative correction of equal magnitude in the opposite (antipole) direction, while the correction for all other parts of the sky is scaled by the cosine of the angle relative to the pole direction.

densities of the points. (This is true for any pair of points with the same r along the line of sight for $r < r_0$.)

For this reason, Oort initially restricted his analysis in 1954 for the $r > r_0$ regions of the Milky Way. As for how he subsequently managed to map out the interior regions of the Milky Way, that is for you to find out by reading his original papers.

Part IV Let's Go Line Fishing [7 marks]

Now that we have a neat little formula on our hands, it wouldn't be fun if we didn't try our hand at tabulating some values to see if it works, would it?

After processing the raw data collected by the radio telescopes, Oort produced a series of what are called line profiles (Figure 4). These line profiles represent the spectral power density at each line of sight (indexed by the longitude) for different frequencies. Here, the data has already been nicely converted from frequency to the corresponding radial velocity according to the Doppler shift formula.

Note that Oort used an older system of galactic longitudes common during his time (defined by the variable l in his paper), which set $0°$ at the intersection between the equatorial and galactic planes. In this system, the galactic centre is at $\theta = 327.5°$ instead of $0°$ as assumed in the derivation in the previous section.

Notice that for the line profiles with longitudes $35°$ to $50°$, there is a very tall and distinct signal peak on the right side. Intuitively, there must be some kind of density structure in the galactic plane corresponding to this peak.

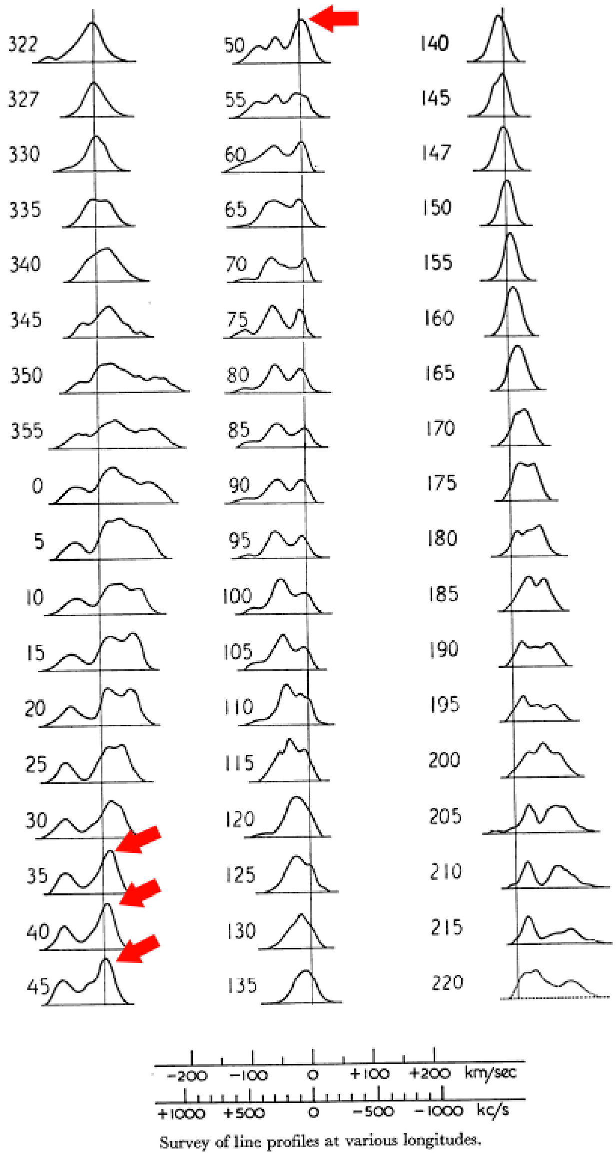


Figure 4: Survey of line profiles at various longitudes from Oort (1954).

In order to map the spatial location of this signal peak, we need to know what is the radial velocity associated with it.

(h) Refer to the table of values in Figures 5 and 6 on the next page. For $\theta = 35°, 40°, 45°, 50°$, find the corresponding radial velocity of this signal peak. [2]

Solution:

The values are given as follows:

θ	v_r
35	+10
40	+5
45	0
50	+5

Line profiles at various galactic longitudes.
The numbers give the measured intensities for the radial velocities V (in km/sec) shown at the top.

l \ V	−120	115	110	105	100	95	90	85	80	75	70	65	60	55	50	45	40	35	30	25	20	15	10	−5	0	+5
60	6	10	12	15	19	22	24	26	28	32	36	41	47	52	57	61	59	52	44	39	39	44	53	64	71	64
55			1	3	6	11	16	23	30	35	38	36	33	34	40	47	53	47	36	34	40	50	56	56	54	51
50				0	3	8	14	20	27	34	39	40	34	31	37	50	58	53	42	34	35	48	66	87	97	100
45						3	11	24	40	50	54	53	47	40	33	29	30	34	44	48	51	58	75	95	105	102
40						2	6	12	20	35	47	53	47	39	30	21	17	18	23	30	38	48	60	76	93	103
35							1	10	19	32	42	46	46	42	32	23	16	15	17	22	28	37	49	63	80	93
30					0	3	7	13	20	27	34	42	43	39	32	24	18	14	16	21	27	32	37	45	56	69
25							1	5	10	17	29	40	43	42	34	25	17	9	7	7	12	22	35	47	59	71
20							3	6	10	15	20	27	35	42	44	42	34	28	23	20	17	17	20	33	51	77

Figure 5: First half of the table of values for negative radial velocity values. Note that the l here refers to the galactic longitude as defined in the old system (which is different from the modern convention θ follows!)

l \ V	+10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	+135
60	42	19	5																							
55	48	33	18	10	4	2																				
50	91	75	49	26	13	7	2																			
45	88	65	39	22	11	4																				
40	97	80	50	27	13	5	1																			
35	99	91	67	45	23	9	3																			
30	80	83	79	74	70	54	35	22	11	3																
25	78	78	76	79	82	82	71	52	34	21	12	5	1													
20	85	82	76	71	68	67	75	82	82	77	61	33	23	13	7	2										

Figure 6: Second half of the table of values for positive radial velocity values.

Next, we want to determine r for a given pair of radial velocity V_r and longitude θ. But first, we need to pick a representative velocity curve for the Milky Way. Luckily, Oort himself provides the one used in the paper (Figure 7).

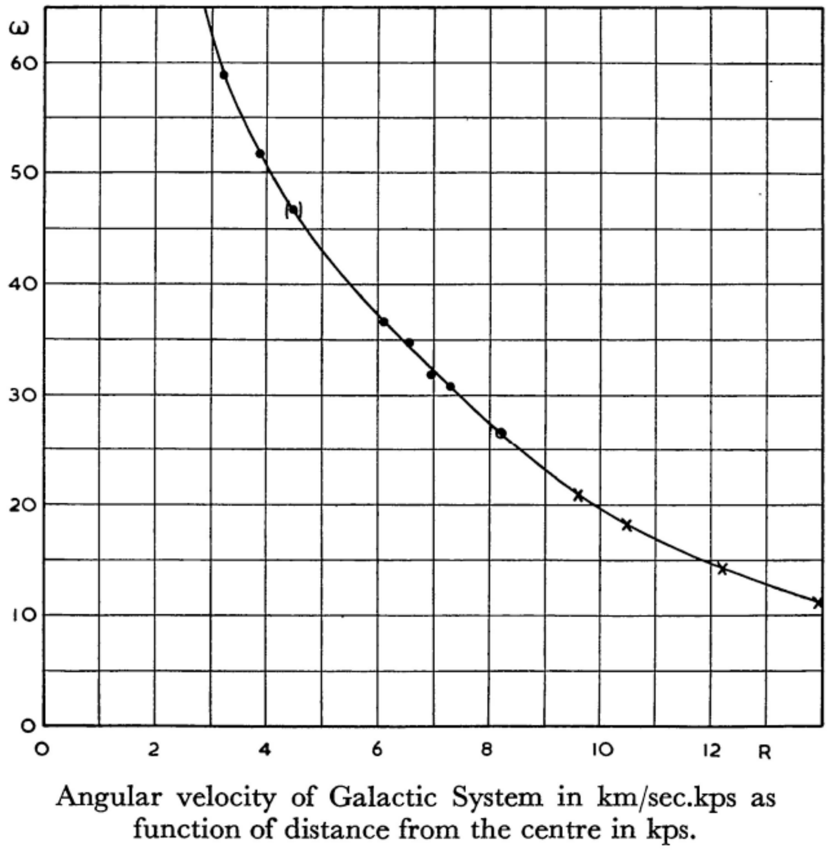


Figure 7: Angular Rotation curve from Oort (1954). Note that the units sec and kps stand for seconds and kiloparsecs respectively.

(j) With reference to the graph of the rotation curve, estimate the orbital angular velocity of the Solar System $\omega(r_0)$ in units of $\mathrm{km\,s^{-1}\,kpc^{-1}}$. [1]

> **Solution:**
>
> The distance of the solar system from the galactic centre is 8.2 kpc, which can be read from the chart in the appendix. This yields an angular velocity $\omega(r_0)$ of around 27 ± 2km s^{-1} kpc^{-1}.

(k) Using the equation for radial velocity V_r, calculate the corresponding $\omega(r)$ of the signal peaks, for $\theta = 35°, 40°, 45°, 50°$. Therefore, estimate the corresponding distance from the galactic centre in kpc. [2]

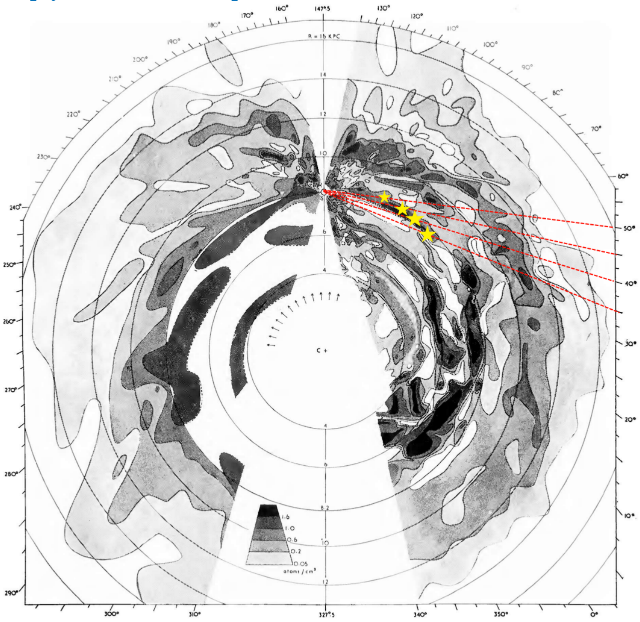
> **Solution:**
>
> After tabulating the values of $\omega(r)$, the corresponding distance r can be read off the angular velocity curve. Due to the large margins of error involved in reading values from a graph, marks are generally given for correct method rather than accurate answers. Remember to add a correction of $+32.5°$ to θ to convert to the correct longitudinal coordinates.

θ	$\omega(r)$	r/ kpc
35	$27 + \dfrac{10}{8.2\sin(35+32.5)} = 28.3 \pm 2$	7.7 ± 0.5
40	$27 + \dfrac{5}{8.2\sin(40+32.5)} = 27.6 \pm 2$	8.0 ± 0.5
45	$27 + \dfrac{0}{8.2\sin(45+32.5)} = 27.0 \pm 2$	8.2 ± 0.5
50	$27 + \dfrac{5}{8.2\sin(50+32.5)} = 27.6 \pm 2$	8.0 ± 0.5

(m) A full-sized copy of Figure 2 is reproduced in the appendix on page 15. On the appendix figure, mark and label the 4 physical points corresponding to the signal peaks for galactic longitudes $l = 35°, 40°, 45°, 50°$ with a cross (beware that this is the old convention!). Therefore, explain if they correspond to a physical structure in the Milky Way, and if so what kind. [2]

Solution:

Given that the tabulated values of orbital radius about galactic centre are close to 8.2 kpc, we can simply trace the 8.2 kpc circle and find its intersection with the corresponding longitudes [1].



Part V All Models are Wrong (but...) [4 marks]

Like any model in science, the model of the Milky Way described above is necessarily simplified for the sake of analysing observational data. Thus, there are certain caveats and uncertainties in the model which limit the conclusions we can draw from the data. For example, the model is limited by the accuracy of the distance of the galactic centre from the Solar System. Oort himself used distance measurements to RR Lyrae variable stars in the Large Sagittarius Star Cloud [3], which is an unobscured region of the Milky Way's central bulge.

(n) Suggest a reason for uncertainties in our distance determinations to the galactic centre using RR Lyrae variables. [1]

[3] van de Hulst, H. C., Muller, C. A., & Oort, J. H., 1954, BAN, 12, 117

Solution:

Even if the Large Sagittarius Star Cloud is not completely obscured, there is still a significant and uncertain amount of extinction due to interstellar dust. This introduces uncertainties in the distance modulus of RR Lyrae Variable [1].

OR

The distribution of RR Lyrae in space is meant to approximate the spatial distribution of the galactic bulge, from which the galactic centre is inferred. However, as RR Lyrae variables on the far side of the galactic bulge are dimmer due to distance and interstellar extinction, there are less likely to be visible. This introduces a sampling bias in the spatial distribution of RR Lyrae variables skewed towards the near side of the galactic bulge [1].

The most prominent feature in Figure 2 are the blank two-sided cones centred about the position of the Solar System. These do not represent regions of insufficient data; rather, the uncertainties in observations were far too large for Oort to deduce the density distribution of hydrogen in these regions.

(o) Explain why was Oort unable to accurately deduce the density distribution in the conical regions. [2]

Solution:

Matter in these regions is travelling in a nearly tangential direction relative to the solar system-galactic centre axis [1]. This means that the radial component of its velocity relative to the solar system is very small, and thus easily masked by observational uncertainties [1].

As mentioned in the previous section, the galaxy rotation curve is merely an approximation of how matter in the galactic plane orbit the galactic centre, therefore this necessarily introduces uncertainties in our model of radial line-of-sight velocities of gas clouds.

(p) Suggest a reason to why the galaxy rotation curve is only an approximate description of how matter orbits the galactic centre. [1]

Solution:

Rotation curves only measure the average orbital velocity of matter orbiting the galactic centre. Individual star clusters or gas clouds may have their own peculiar velocities that deviate significantly from this average orbital velocity [1].

OR

The rotation curve assumes that matter orbits the galactic centre in circular orbits. However matter in the galaxy may orbit the galactic centre in significantly elliptical orbits instead [1].

The uncertainties and limitations of the model thus led Oort to caution that the numerical densities of the gas cloud he obtained are highly uncertain, and need to be backed up by further data and observations. Nevertheless, his observations were remarkably accurate, and his methods were further refined with new observations to produce more and more accurate maps of the Milky Way.

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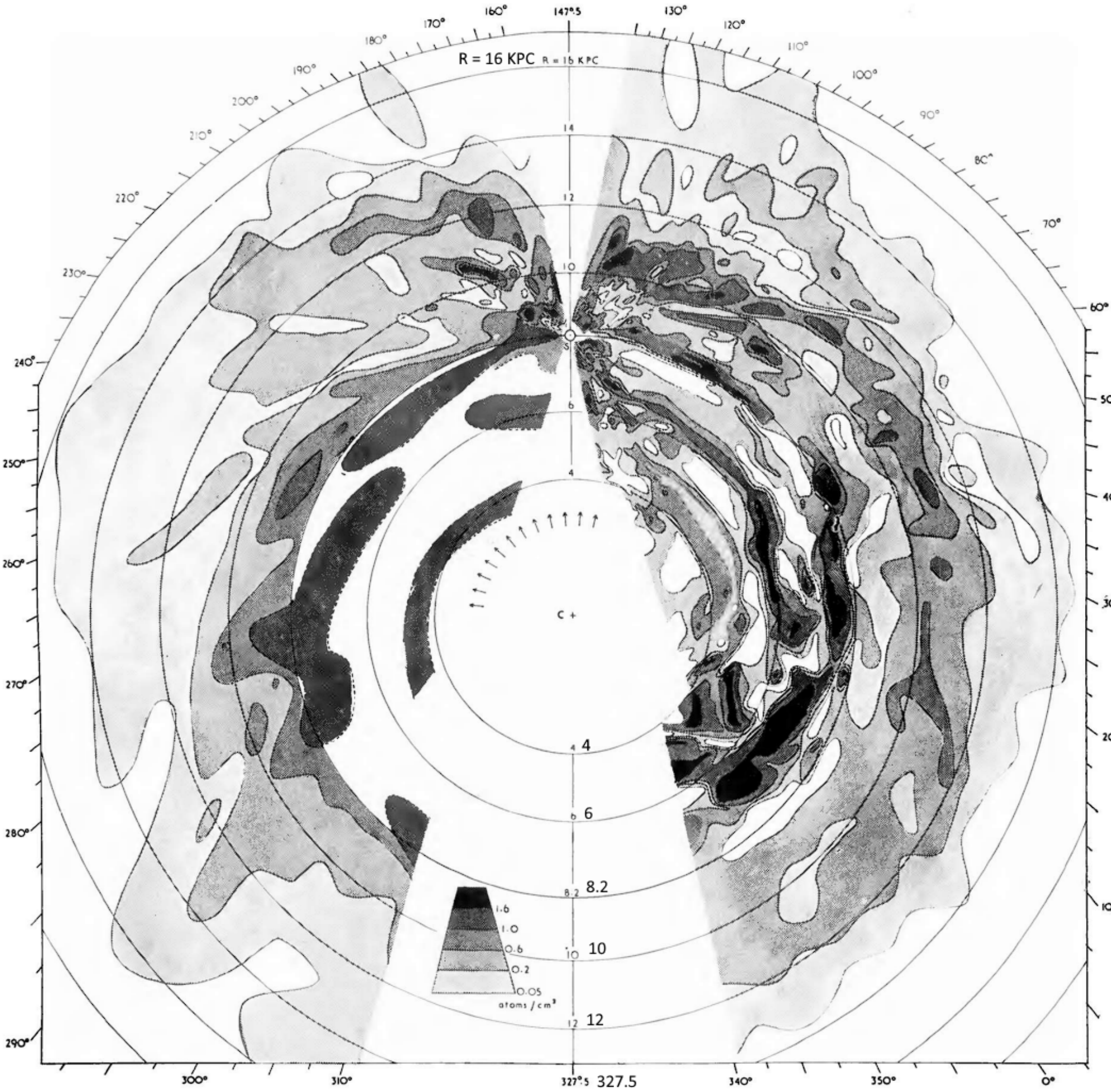
Part VI Appendix



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Question 3 MOND-ifying Newton's Gravitation [20 marks]

Disclaimer: While you embark on this absolutely amazing question, I would advise you to **not get scared by words**. If nothing else, I hope to impart the following takeaway:

Scientists love to make new theories, it is hard to prove those theories, and the nature of science is VERY MESSY.

Part I Why do we Scrutinize Newton? [2 marks]

Newton's laws of motion are arguably one of the most famous set of physical laws. They are usually the first physical laws we learn in a physics class. We spend so much time solving questions using these laws. Obviously, they must be correct right? Well, maybe not.

There have been multiple occasions where Newton's laws have failed to accurately describe physical phenomenon. Some of you might be familiar with Einstein's theories of special and general relativity, where we must modify Newton's laws to accurately predict physical parameters.

Another challenge to Newton's laws, one which is very famous in astronomy and astrophysics, comes from galaxies. Figure 8 below shows a galactic rotation curve. On the y-axis, we have the rotational velocity of stars about the galactic centre, and on the x-axis, the distance from the galactic centre.

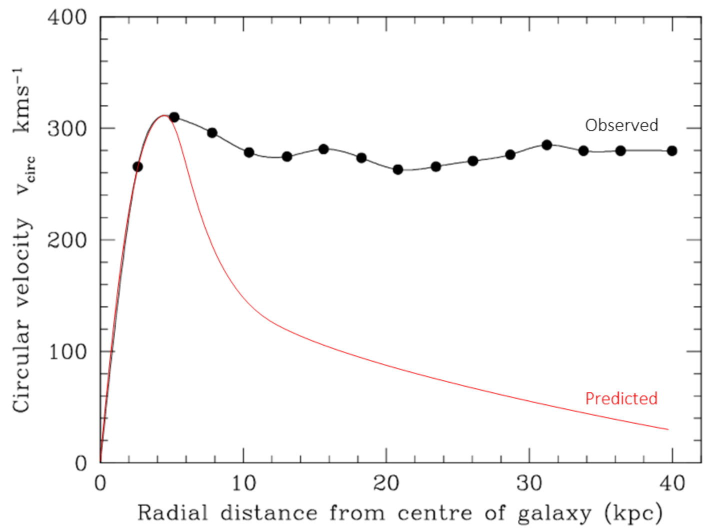


Figure 8: The galactic rotation curve. The red curve shows the predicted velocity, whereas the black curve is the observed.

Thus, this curve shows the rotational speed of a star when it is at a certain distance away from the centre of the galaxy. As is blatantly obvious in this figure, the predicted curve fails to match the observational data. **The red curve is plotted based on Newton's universal law of gravitation** (which can be found in the Formula Booklet) **and Newton's second law.**

(a) State and explain <u>**ONE**</u> possible reason for the observed data deviating from the predicted curve. [2]

Solution:

There is extra mass in the galaxy which is not accounted for [1]. As per Newton's Law of Gravitation, the greater the mass the greater the centripetal force on the star, and thus, a greater rotational speed of a star.

If the observed speed is greater than predicted, some mass in the galaxy is not accounted for [1]. This "hidden mass" is called Dark Matter, theorised to be present in Halos around the Galaxy (additional info; not necessary to obtain the mark)

Apart from what you guessed above, there is a much more pessimistic answer – Newton's second law is wrong! I would be stunned if somebody tells me $F = ma$ is incorrect, but that is what astrophysicist M. Milgrom conjectured in 1983. He said Newton's second law is not completely correct and proposed the **modified Newtonian dynamics (MOND)** to account for differences in the predicted and the observed curve. For the rest of this question, we will explore MOND and the techniques used thus far to try and prove its validity.

Part II Changing the Second Law?! [5 marks]

Now, this is when things will start getting uncomfortable. In its most used form, Newton's second law is

$$F = ma$$

However, MOND modifies the above equation to

$$F = \mu ma$$

where μ is a **function** dependent on acceleration a. He further defined two acceleration regimes:

1. the high-acceleration regime where $a \gg a_0$, and
2. the low-acceleration regime where $a \ll a_0$.

where a_0 is a **constant** with $a_0 \approx 1.2 \times 10^{-10}\,\mathrm{m\,s^{-2}}$. Before we move any further, let us get a sense of scale here, and see how small, or big, this value of a_0 is.

(b) Using data from the Formula Booklet, calculate the centripetal acceleration of Earth around the Sun (ignore the effects of MOND). [1]

Solution:

Using the formula of centripetal acceleration and plugging in the necessary values,

$$a_c = \frac{v^2}{r} = \frac{(2\pi r/T)^2}{r} = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 \times 1.496 \times 10^{11}}{(365.24 \times 24 \times 60 \times 60)^2} = 5.93 \times 10^{-3}\ \mathrm{ms^{-1}}$$

This is so much larger than a_0! Accelerations that approach a_0 are found on much larger distance scales (which we will explore very soon). For now, we will focus on the MOND version of Newton's second law. While the function μ has various forms, let us consider its two features:

1. in the high acceleration regime where $\mu \approx 1$, and
2. in the low acceleration regime where $\mu \approx a/a_0$.

In the high acceleration regime, MOND gets reduced to the standard Newton's laws of motion as

$$F = \mu ma = 1 \times ma = ma.$$

Here, we expect and can empirically show that the standard Newton's second law is a good model for scales within our Solar System such as in part **(b)**. Therefore, MOND has to be able to produce similar results in high acceleration regimes. However, the story changes when we enter the low acceleration regime.

(c) Write Newton's second law in MOND's low acceleration regime in terms of a, a_0 and m. [1]

> **Solution:**
>
> In the low acceleration regime, $\mu = a/a_0$. Therefore,
>
> $$F = \mu ma = \frac{a}{a_0} \times ma = m\frac{a^2}{a_0}$$

(d) Using your result in part **(c)** along with the Newton's universal law of gravitation and centripetal acceleration, find an expression for the rotational speed in terms of G, a_0 and M where M is the mass of the galaxy. You may assume that the mass of the galaxy is concentrated in the centre of the galaxy with a star (of mass much smaller than the galaxy) orbiting around it. [2]

Hint. *The centripetal force is provided by the gravitational force.*

> **Solution:**
>
> As the centripetal force is being provided by the gravitational force, and centripetal acceleration is $a = v^2/r$, we can use Newton's law of universal gravitation to find the rotational speed.
>
> $$\frac{GMm}{r^2} = m\frac{\left(\frac{v^2}{r}\right)^2}{a_0} \Rightarrow \frac{GM}{r^2} = \frac{v^4}{r^2 a_0} \Rightarrow v^4 = GMa_0$$

(e) Hence, explain how rotational speed as predicted by MOND (your answer in part **(d)**) explains the plateau in the galactic rotation curve as shown in Figure 8. [1]

> **Solution:**
>
> As observed in the relationship above, the rotational speed is independent of the distance from the galactic centre, r (in the low acceleration regime). This means as the distance increases, the rotational speed remains constant. Hence this explains why we observe a plateau in Figure 1.

Well, MOND can explain why we see this plateau, so it must be correct, right? Not necessarily. We can find MOND regimes in other parts of this vast universe as well. As always, if a theory accurately predicts one scenario, it should work in other similar scenarios as well. Otherwise, it is useless.

Part III WB (not Warner Brothers) and GAIA (not the one in Percy Jackson) [7 marks]

How do we test if MOND works? We look at accelerations in the MOND regime. It is very important to note that while Part II of this question exposes us to two forms of the function μ, it has more forms. **Thus, MOND's regime in which it deviates from Newtonian mechanics occurs more generally when $a \leq a_0$ and NOT only when $a \ll a_0$.**

Another place where people go to check MOND is by looking at *wide binaries*. Now some of you might be familiar with Binary stars such as Sirius A and Sirius B, Alpha Centauri A and Alpha Centauri B, etc. A binary star system is a system of two stars that revolves around a common centre of mass (commonly known as the barycentre).

Now, **wide binaries** are binary stars that are separated by a "large" distance. The definition of "large" is not set in stone but it is generally greater than multiple thousand Astronomical Units (AUs). At such large separations, the acceleration between the stars becomes less than or equal to a_0 and hence, they enter the MOND regime. Moreover, these distances are still "small" when compared to galactic scales (for an idea of scale, the diameter of the Milky Way is approximately 5.7×10^9 AU).

Additional Info. *At such "small" scales, dark matter (**hint** for one of the previous questions), if it exists, is so diffused that it should not have any effect on the binary star. Thus, any increase in speed could be explained using MOND.*

Well, there is one *small* issue with wide binaries. **We cannot just "look" at two stars, wait for them to complete one orbit around a common centre of mass, and call them a wide binary.** Let us consider an example to see why is this the case. Figure 9 shows a hypothetical wide binary system.

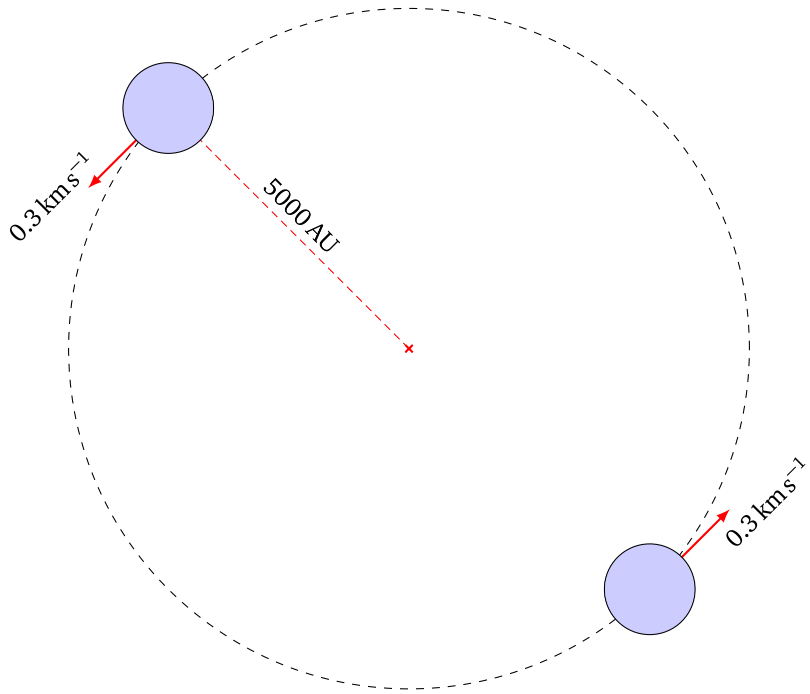


Figure 9: A hypothetical wide binary system in which two stars of the same mass are orbiting each other with a semi-major axis of 5000 AU in a perfectly circular orbit and with an orbital velocity of 0.3 km s^{-1}.

In this system, we have two stars of the same mass rotating around the barycentre in a perfectly circular orbit. The semi-major axis is 5000 AU, and the tangential speed of each of the star is 0.3 km s^{-1}.

(f) Explain why we cannot directly observe a full orbit of the star system shown in Figure 9. [1]

Solution:

Option 1 → The orbital period of the system above is,

$$P = \frac{2\pi R}{v} = \frac{2\pi \times 5000 \times 1.496 \times 10^{11}}{0.3 \times 10^3} = 1.5667 \times 10^{13} \text{ s} \approx 500000 \text{ years}$$

which is too long for mankind to observe directly.

Option 2 → (not intended but will accept): External gravitational perturbations could destablize the wide binary and we might not be able to see the binary complete its orbit.

Well, since looking at such stars directly is not an option, we need some other way to check whether MOND works or not using these Wide Binaries. How hard can this be? We can break it down in two *easy* steps (**hint** for a question far below):

1. we need to confirm that star systems are indeed wide binaries, and
2. we need to find few parameters, such as the orbital velocity of the binaries, the separation of the stars in the binary, and the masses of the binaries to test MOND.

Multiple methods can be employed to get answers to the above two steps. However, in this question, we will limit our discussion to what **Banik and his buddies** did in their recent study. How did they even do step 1 if we can't map the orbits of such stars?

This is where we need to introduce *Gaia*. Gaia is a spacecraft launched by European Space Agency back in 2013, and is providing us with extremely accurate data about objects in the night sky. This data includes both astrometric (position measurement) and photometric (brightness related measurement) data. Going over every data set collected by Gaia is way beyond the scope of this question but let me introduce you to some important parameters.

The most *obvious* thing we need to check is whether the two stars are at a similar distance. For this we use something called the *parallax*. When Earth is on one side of the Sun, Gaia looks at a star with respect to the background stars. Approximately six months later, when the Earth is on the other side of the Sun, Gaia looks at the same star again and sees how much it has shifted. Since we know the distance between Gaia and the Sun, and now we know one angle in this triangle, we can calculate the distance to the star easily using basic trigonometric principles. The *parallax angle* is defined to be angle α shown in Figure 10. Refer to the formula Booklet for a nice estimation of parallax and distance.

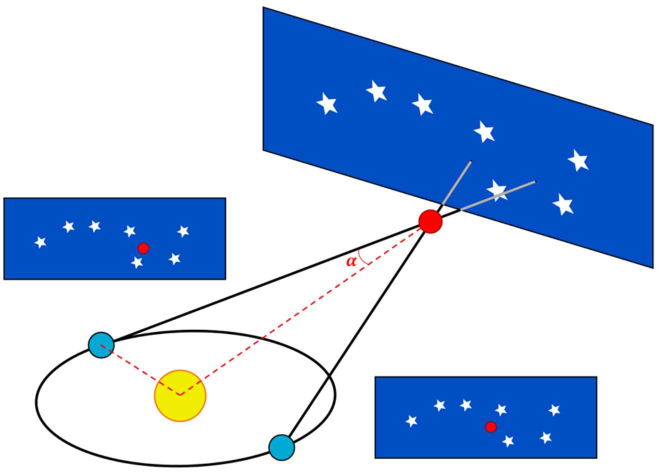


Figure 10: Diagram of Parallax (Source: Wikipedia)

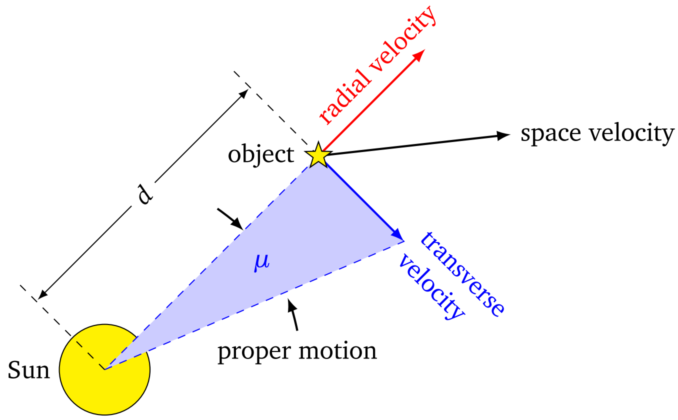


Figure 11: Diagram of proper motion.

The second parameter which is very necessary here is *proper motion*. While we all say the stars on the celestial sphere are *fixed* and they move because of the Earth's rotation, that is not completely true. In fact, the stars do move on the celestial sphere, and we call this motion *proper motion*. Our Solar System revolves around the centre of the Milky Way. However, this revolution is very slow, with a revolution period of about 230 million Earth years! Therefore, the magnitude of proper motion is very slow. Figure 11 above is a good representation. The star can have some actual velocity, denoted in Figure 11 by the black arrow. We can decompose it into the radial velocity as shown in red, which denotes the motion along our line of sight, and the transverse velocity as shown in blue, which denotes the motion perpendicular to our line of sight. Proper motion is the **angular change** of the position of the object in the direction of its transverse velocity. It is measured in arcseconds per year (as yr^{-1}) or milliarcseconds per year (mas yr^{-1}).

Combining proper motion and parallax yields the transverse velocity of the stars!

Banik and his buddies said: we accept a wide binary pair if the difference between the transverse velocity is less $3\,km\,s^{-1}$.

(g) Reading this statement, my friend Nuxix says "This is wrong. The transverse velocity difference should be $0\,km\,s^{-1}$ because wide binaries are gravitationally bound and will appear to move together." Is she **COMPLETELY** correct? Explain. You may use any diagrams to support your answer. [3]

Solution:

She is not completely correct (will accept incorrect as well). While it is correct that they are gravitationally bound [1], it is a binary star system, and the stars are moving in opposite directions with respect to each other. So even though their transverse velocity might be low, there will definitely be a difference in the relative transverse velocity due to motion in different directions [2].

(Extra information: The uncertainties in Gaia data makes it necessary for us to choose a range of accepted values.)

Now since we know the distance to the stars, and Gaia also knows the positions of the stars in the night sky, we can also calculate the distance between the stars. The only thing left is mass which...

(h) While writing the paragraph above, Iur Nuj sitting next to me looked at the question and interjected, "Isn't this easy? Just use Newton's law to find it out, same as what we learnt in school. Briefly explain why Iur Nuj is very wrong. [1]

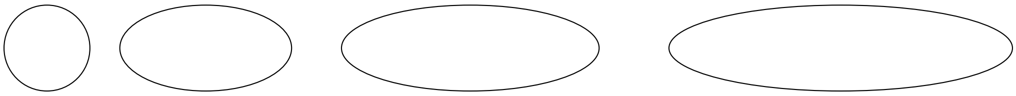
Solution:

We are trying to test whether MOND is correct, and MOND modifies Newton's law. Therefore we cannot use Newton's Law to calculate the mass because we don't know if it is correct to begin with [1].

The mass is thus calculated using empirical relationships between photometric magnitudes and distance. We will not delve deep into the mass discussion in this question.

Banik and his buddies should have just stopped here, right? They already have the velocity and separation data. Why not simply plot a graph of velocity against separation to verify the centripetal acceleration? One might argue, "Gaia data now has very little uncertainty, so maybe our methods are flawed," but that would be unwise. There are numerous parameters and unknown sources of errors that we are completely unaware of.

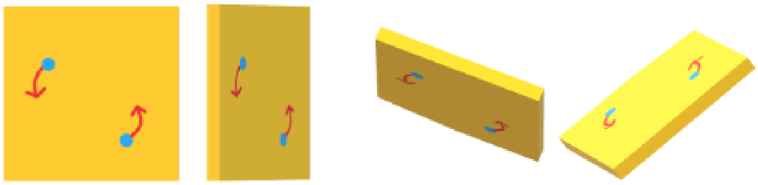
For example, how eccentric is the orbit of a wide binary? In simpler terms, how much of an oval is the orbit? Is it like Figure 12A, 12B, 12C, 12D, or something else?



A B C D

Figure 12: A to D from left to right, in increasing order of eccentricity of the orbit.

Moreover, what is the inclination of the orbital plane of both stars? Suppose the two stars orbit lie on a piece of paper. What is the angle at which the piece of paper faces us? Is it something like Figure 13A, 13B, 13C, 13D, or any other angle?



A B C D

Figure 13: A to D from left to right. The yellow cuboid is the orbital plane, the blue dots are the stars, and the red arrows denote the direction of rotation.

(j) Apart from eccentricity and inclination, state and explain one more factor which we do not know. [2]

Hint. *There is a lot which we don't. To figure out one of them, think about the closest star system to the Sun.*

Solution:

Some possible solutions ([1] for stating and [1] for correctly explaining):

1. Presence of Close binaries: In some binaries, a third star might be close to one of the Wide Binary stars. Gaia might not resolve it; hence, we will not know its gravitational effect on the binary. Also, we might end up underestimating the mass of the system during our analysis.
2. Line of Sight contamination: The light reaching gaia could have been affected by other sources (gravitational lensing, extinction due to dust, etc) which will have an affect on the measured variables (however, this effect is expected to be negligible in Gaia).

The list above is non-exhaustive and there are other unknown variables as well.

However, things are not as bleak as you may think. We can use a very cool tool to solve this, which I am sure all school students love: *statistics and probability*.

Part IV Welcome Prob-ability [4 marks]

While all our efforts thus far might seem futile, scientists can always figure out ways around things. As we do not know the distribution of many parameters mentioned in the previous part, we can create *probability laws* to predict what these parameters could likely be.

For those of you unaware what probability is, it is a measure of “chance” or “likelihood” of something happening. It can be found by dividing the expected outcomes (what you want) by the total set of outcomes (all the possible outcomes). For example, the probability of getting a head in a *fair* coin toss is 1/2, because there is one outcome that we want (heads) and there are two outcomes in total (heads and tails).

Now, instead of considering discrete cases (like heads and tail), we can consider continuous values of parameters like eccentricity (0 to less than 1) and plot a graph for it. **The more likely a certain outcome, the higher will be the graph at that point**. For example, in Figure 14 below, the most likely eccentricity of a wide binary is 0.50. If you want to find what is the probability that the eccentricity is between two numbers, let’s say 0.75 and 0.80, you just need to find the area under the curve (the red dashed area).

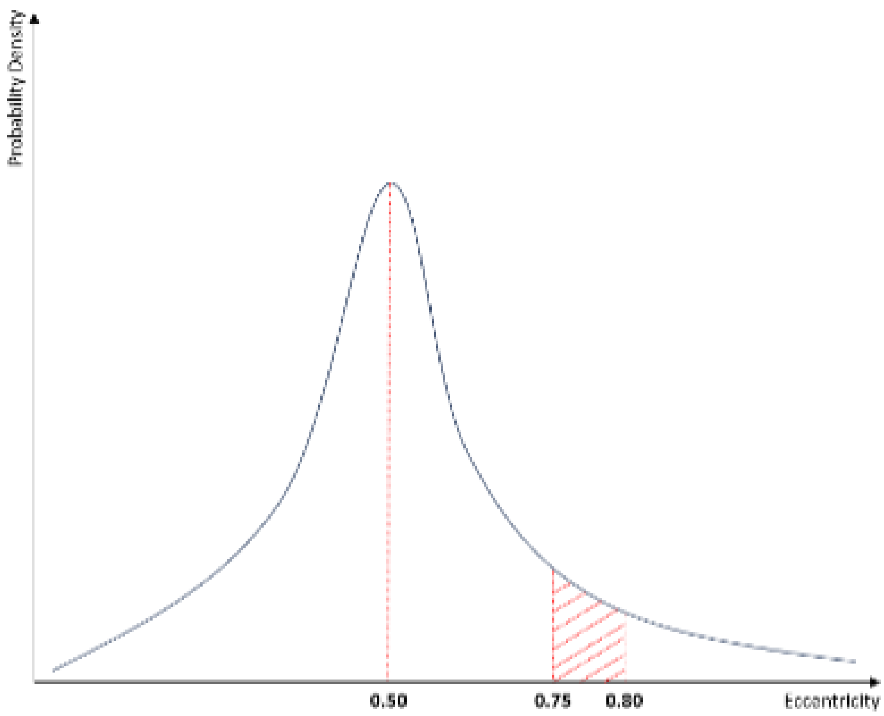


Figure 14: Probability distribution function (in blue) for eccentricity.

Banik and his buddies performed such probability analysis on **seven different variables** using their dataset of wide binaries and found the **most likely value** for each of these parameters.

The **most important** parameter for us is α_{grav}. It is a parameter Banik and his buddies introduced to account for gravitational force. **If the value of α_{grav} is 0, the acceleration in wide binaries is completely Newtonian**

(no effects of MOND whatsoever). If the value is 1, the acceleration is exactly what MOND would predict. Figure 15 shows the probability of distribution of α_{grav}.

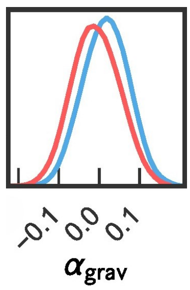


Figure 15: Probability distribution function (the red curve) for α_{grav} considered by Banik and his Buddies. **Ignore the blue curve** (It is beyond the scope of this question).

(k) Based solely on Figure 8, do wide binaries obey Newtonian gravity or MOND? Explain your answer. [2]

Solution:

As observed in the probability distribution function of α_{grav}, the **most likely value** of α_{grav} is 0.0 because of the peak of the distribution is very close to 0.0 [1]. This suggests that the gravitational law which WBs follow is most likely Newtonian [1].

However, if you still remember, Banik and his buddies also calculated the velocities and the separation between the stars. Plus, if their research paper is 48 pages long, they must have done multiple analysis of their dataset. Thus, Figure 16 shows their analysis on the orbital velocity and the separation between the Wide Binaries. The pink bars are the observed data. They created two models, one for Newtonian (in black) and one for MOND (in Blue) and plotted them on the same graph.

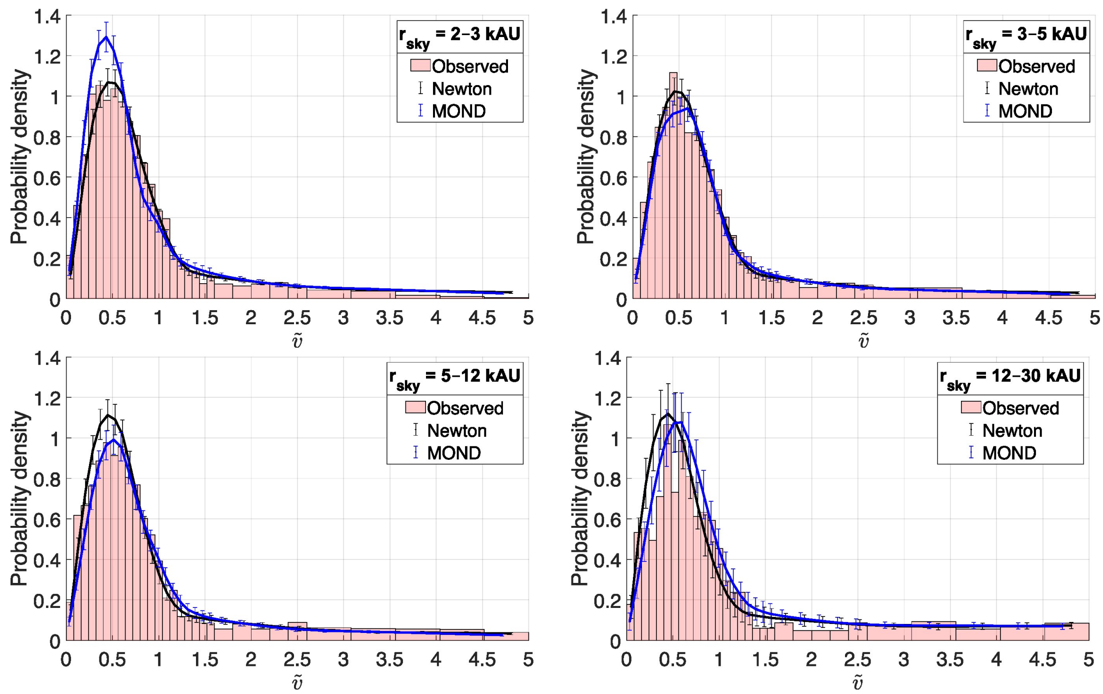


Figure 16: Probability distribution function for different separation of Wide Binaries (r_{sky}) and their orbital velocities ($\tilde{v}$). The black line is the Newtonian Gravity model, and the blue line is the MOND model.

(l) Based solely on Figure 16, do wide binaries obey Newtonian gravity or MOND? Explain your answer. [2]

Solution:

The results seem inconclusive solely based on this figure [1]. In the 2-3 kAU region, Newtonian model seems to fit the data better whereas in the 5 - 12 kAU region, MOND does. In the 3-5 and 12-30 kAU region, it is very hard to tell which model's fit is the best. Thus, we cannot conclude whether MOND is preferred over Newtonian gravity [1].

If I have asked you two questions, the answer should vary between the two (otherwise what is the point). As you have observed, slightly different analysis of data, by the same team, could show different statistical results.

Part V Science is very Messy [2 marks]

Now if you were wondering why I choose MOND out of all the theories that exist, it was to think about how astronomy is not straightforward at all. Earlier in this question, you were exposed to the Gaia satellite. When Gaia released its latest dataset (Data Release 3), there was excitement because a lot of science could be done on it. This dataset was used by Banik and his buddies (which we have discussed in this question). With the data release being publicly available, multiple other research groups performed their analyses, and reached drastically **different conclusions**!

Another researcher, KH Chae used the same idea of finding wide binaries and trying to test whether MOND or Newtonian gravity is preferred. Figure 17 shows comparison of Chae and Banik and his buddies' datasets.

On the y-axis, we have the orbital velocity (median $\tilde{v}$) of the wide binaries and on the x axis, we have the Wide Binary separation (median r_{sky}/r_M). The grey dashed line in both graphs represents the predicted observations if MOND was correct.

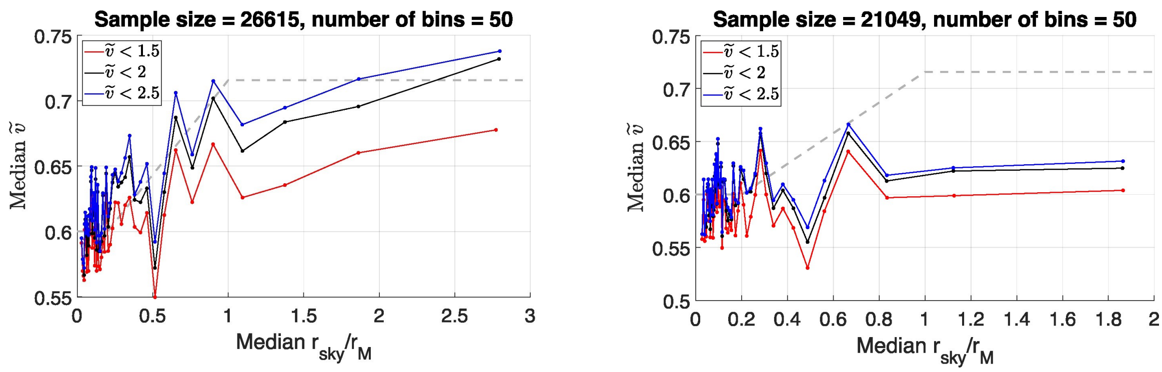


Figure 17: Chae's dataset (left) and Banik's dataset (right). The sample size denotes the number of wide binary pairs in their own analyses. *You may ignore the number of bins.*

(m) Why are the graphs in Figure 17 of results from both research groups, significantly different? [2]

Hint. *If stuck, go back to the two easy steps mentioned in Part III.*

Solution:

QM's potential answer: From Figure 10, we notice that Chae's sample size is larger than Banik and his buddies' sample [1]. This could imply both groups used different data filtration processes. Given the larger sample size, Chae must have had less stringent conditions on what passes off as a wide binary [1]. Hence the difference in graphs.

The fact that each research group also have very bold claims makes it even confusing. Banik and his buddies said that the chance of their conclusion being a mere statistical fluke, is 1 in 10,000 trillion, which Chae says 1 in 2 trillion!

However, we cannot really conclude who is *right* over here. I have just exposed you to the world of scientific drama and MOND. Go ahead, read more literature, and you can make your own logical conclusions. My job was just to point you in a direction.

Statistics in astronomy (and science generally as well) is usually like this. To quote Ernest Rutherford,

"If your experiment required statistics, you ought to have done a better experiment."

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Question 4 Cepheid Conversation [20 marks]

Introduction

A chilly winter breeze blows across your face as Max and his three friends set up for the night at the Torrance Barrens Dark Sky Reserve. The trees have shed their leaves and a layer of snow blankets the ground. In the western sky, Max makes out a young waxing moon—young enough that it looks sharp enough to cut through the skies. To the east, the belt of Venus displays a muted pink whilst sitting atop a band of navy blue. Night is approaching.

“I hope the skies remain this clear for our observation of the Geminids tonight,” Max mutters under his breath.

As night slowly falls and the stars start their show, more people arrive. Soon, your group is accompanied by ten other groups of people, all trying to enjoy the night sky together.

“So I heard that Jupiter is up in the sky tonight. How can we tell which one is Jupiter?” Max’s engineering coursemate Henry asks Max.

Max explains that stars twinkle while planets do not, and that Jupiter is also one of the brightest objects you’ll see in the night sky on that night. Fleming continues to explain that the twinkling is due to the atmosphere before Henry can respond.

“So stars don’t actually twinkle on their own?” Henry continues.

As you’re about to respond, Annie interjects: “Well actually, some do! These stars actually do vary in brightness. But they do it with more regularity than the ‘twinkling’ we see due to the atmosphere. Let’s see if we can spot ome tonight whilst meteor watching!”

Annie pulls up a few tables of these *variable stars*, one of them shows a few examples of different classes of variable stars:

Star	**Variable Class**	**Period (Days)**	**Est. Density** (ρ_{Sun})
Mira A	Mira Variables	332	$\sim 10^{-8}$
η Aql	Classical Cepheid Variables	7.2	$\sim 10^{-5}$
RR Lyr	RR Lyrae Variables	0.57	$\sim 10^{-3}$
γ Boo	δ Scuti Variables	0.29	$\sim 10^{-2}$

Table 1: Examples of different variable stars.

“I notice that some of the class names are the same as the star name!” Henry exclaims.

“Yeap! By convention the first star identified in the class of variable stars will have the class named after it.” Max follows up.

“Oh, so just like children names, the first child is named after you.” Henry cheekily mentions.

“No one wants a mini-you, Henry.” Annie mentions whilst Fleming laughs at the side.

“So these variable stars are important in astronomy because their intrinsic brightness is linked to the length of time between peak brightness. That’s the period that column three refers to,” Max mentions.

“Wait, so how come these stars have such regularity in their luminosities? What sort of magic is going on?” Henry asks.

It is now where the two astrophysics majors look at each other and grin.

"Oh crap, I'm about to get an Astronomy 101 crash course out here, right now aren't I?" Henry mentions with Annie and Fleming turning away from each other and excitedly, nodding their heads.

"Ok ladies, let's make this fast. Astronomical twilight is ending soon and we should focus our attention to the skies after," Max mentions.

Part I Pulses [8 marks]

"The star's brightness varies because it is actually physically changing. You did your thermodynamics course so you should be familiar with radiation," Fleming mentions.

"Haha I wouldn't say *familiar*, maybe I-heard-it-mentioned-in-class-before familiar," Henry scratches his head.

"You just need to know that the power output of a blackbody is proportional to the surface area. Surely you remember that!" Max mentions.

Henry gives Max a sly grin.

"Ok, he spoiled it. So what's happening is that the star is physically growing and shrinking. When it does, the luminosity changes with it. Because the process of growing and shrinking is periodic, so will the luminosity change! The star's size is basically oscillating." Annie continues.

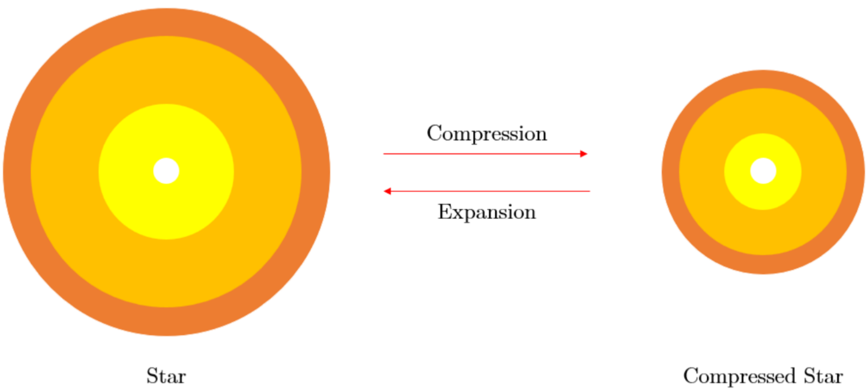

Figure 18: Expansion and contraction of a star.

Sound in Space?

Max thinks about what Annie and Fleming just mentioned and recall back to his class on waves and oscillations. He starts to piece together some more information as Henry, Annie, and Fleming go on a tangent because Henry asked about if these variable stars were cosmic versions of bouncy castles.

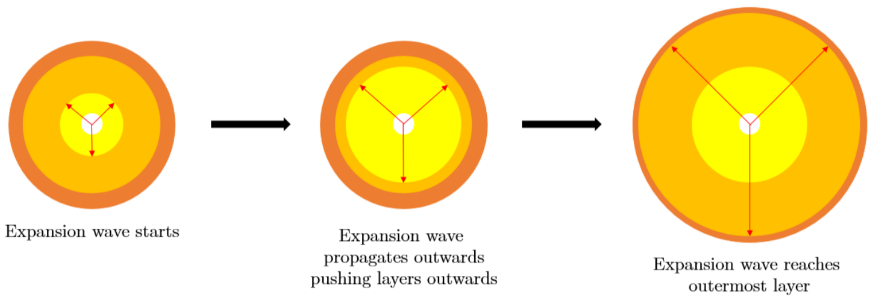

Figure 19: Expansion waves.

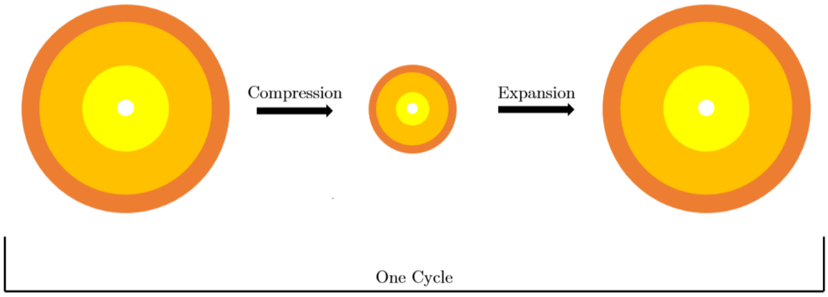

Figure 20: One cycle of compression and expansion.

The star cannot instantly change size, so there must be some form of wave that travels through the stellar material.

"So we basically have a net sound wave that travels within the star. But they must interfere correctly to produce a net change in the star's size," Max realizes.

(a) With reference to these sound waves, what effect is at play here to produce the observed pulsation in stars? [1]

Hint. *What kinds of waves have been set up in the interior of the star?*

Solution:

Resonance has occurred within the star. A *standaing wave* has been set up to produce a net change in the star's radius [1].

Note: Simply stating that constructive interference is not enough as that occurs anything two waves in-phase interact. The constructive interference must produce a standing wave for the star to periodically change size.

Max recalls that the relation between the speed of sound and the density of the stellar material are linked:

$$v_s = \sqrt{\frac{\gamma P}{\rho}}$$

where γ is the specific heat ratio of gas. The pressure P of the stellar material can be found from the hydrostatic equilibrium equation. He quickly Googles for the equation and it returns the equation

$$P(r) = \frac{2}{3}G\pi\rho^2\left(R^2 - r^2\right)$$

where R is the radius of the star.

"Only for constant density stars," Max whispers to himself.

Recalling from earlier that this sound wave needs to travel from the core to the surface, Max fixes the last piece of the puzzle. He's surprised at the result:

(b) Show that the period T of oscillation is independent of the star's radius R. [3]

Hint. *This integral might come in handy but there may be an alternative solution.*

$$\int_{\alpha}^{\beta} \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_{\alpha}^{\beta}.$$

Solution:

Brute force version

The brute force technique is suggested by the hint, to do some form of integral. This integral comes from the oscillation period, remembering the factor of two as we are dealing with oscillation periods:

$$T = 2\int_0^R \frac{dr}{v_s}$$

Combining the equation for the speed of sound and hydrostatic equilibrium, we get,

$$v_s = \sqrt{\frac{2}{3}G\pi\gamma\rho(R^2 - r^2)}$$

We then get:

$$T = 2\int_0^R \frac{dr}{\sqrt{\frac{2}{3}G\pi\gamma\rho(R^2 - r^2)}}$$

$$= 2\sqrt{\frac{3}{2G\pi\rho\gamma}}\left[\arcsin\left(\frac{r}{R}\right)\right]_0^R$$

This gives us,

$$T = \sqrt{\frac{3\pi}{2G\rho\gamma}}$$

The final expression of T does not include R. Hence, we have shown that T is independent of R.

Dimensional Analysis version

This solution involves dimensional analysis. We see all the variables and constants with dimensions that could end up in the expression for T. There are: R, G, and ρ. We ignore γ and π since they are dimensionless (unitless). For the three identified and the period T, they have the following units:

$$[G] \rightarrow \mathrm{m^3\ kg^{-1}\ s^{-2}}$$

$$[\rho] \rightarrow \mathrm{kg\ m^{-3}}$$

$$[R] \rightarrow \mathrm{m}$$

$$[T] \rightarrow \mathrm{s}$$

So we need to find a way to combine G, ρ, R and T,

$$T = kG^a\rho^bR^c$$

where a, b, and c are constants to be determined, and k is some dimensionless proportional constant. We can observe the exponents of the units and then get the constants:

$$a = -\frac{1}{2}$$

$$b = a = -\frac{1}{2}$$

$$c = b - a = 0$$

The most important result is that c is 0. This means T is independent of R. The values point to a relation,

$$T = \sqrt{\frac{K}{\rho G}}$$

which is what we found in the main solution if we set $K = 3\pi/2\gamma$.

"Hey Annie, could you show me that table again about the different classes of variable stars? I wanna confirm something," Max asks, interrupting the trio's intense discussion.

Annie passes him her phone with Table 1 displayed.

(c) Does the previous result align with the observations made given in Table 1? Briefly justify your answer. [2]

Solution:

Yes [1]. From Table (1) we see that the star's oscillation period decreases with increasing density [1] - exactly the same trend we expect from equation obtained in the previous part.

"Ok, so we can do the simplest mode of oscillation, where the star is just pulsating radially only. For this mode there will be different displacements about their original positions at different radial positions from the core," Max mutters as he talks to himself again.

(d) For this fundamental oscillation mode, where would we expect the layers to have maximum and minimum displacements? [2]

Solution:

We expect maximum displacement to occurs at the surface of the star and minimum displacement to be at the centre [2].

This situation is akin to a standing wave being setup in the half-opened tube. We look at the fundamental frequency, where the node is situated at the centre of the star and an anti-node is situated at the surface. In fact, $beta_{min} \approx 0$. Question (a) was meant to be a hint for this question.

"Hey you're spacing out again! Hello? Henry to Max, come in." Henry waves his hand in front of Max, snapping him out of his trance.

Part II What's Driving this Thing? [[12 marks]]

Max returns back to the conversation. It seems like the conversation has shifted to how the star is getting its energy to grow and shrink.

"So is it like heat engine where the compressions and expansion of the stellar gas produces a net positive work?" you ask.

"Well, not quite. You're thinking along the lines of periodic energy generation. Let's see why it cannot be what you say." Annie mentions.

Periodic Energy Generation

"So the work done by a gas is the sum of the gas pressure times its volume change over the whole cycle right?" Annie continues.

"You can say $\oint P\,\mathrm{d}V$ Annie. They're not JC students who are afraid of integrals." Fleming interrupts.

"Ok sure, I was just making sure Henry's following."

"Anyway... the work done is just the sum of $P\Delta V$. Mathematically, that means that there must be a nonzero ΔV occurring at a nonzero pressure for work to be done. The pressure is greatest at the core but..." Annie continues.

It's at this moment when Max realises why Annie mentioned that such oscillations would not be the main driver of the oscillations in stars.

(e) For the mode of oscillation that Max considered previously, suggest why the current proposed model cannot be the main source of energy driving such types of oscillations near the core. [2]

Hint. *Consider back to how the star is oscillating. It might help to compare this kind of oscillation to a sound wave in a tube.*

Solution:

The hint to this question comes from the previous question where we were asked for the location of the max and min displacements. Where there is displacement ($beta$), there will be compression/expansion of the gas. There is very little compression near the core for our discussed mode of oscillation. Thus,

$$\Delta V \propto \beta^3$$

This implies that the work done will be tiny and close to zero near the core - definitely insufficient to drive such massive pulsastions in the star.

Period Energy Release

"Ok, spare me the math, just tell me how these stars work." Henry exclaims.

"Ok, so... it's not a periodic energy generation that causes the star to change size. The star is basically bottling up its energy and releasing it periodically. When the star bottles up more energy, it grows, then releases it and shrinks. Rinse and repeat," Fleming explains.

"How does a star bottle up its energy? It does not like, have feelings or whatever right. So what, it reabsorbs the light it produces?"

"Even a stopped clock works twice a day but yeah, you're right," Annie mentions.

Henry scoffs.

"Ok yes. So when the star shrinks, it gets more opaque and absorbs enough light and energy to heat the gas and make it expand. To describe it fully we need to lay some groundwork," Fleming mentions.

"We'll start by treating the stellar gas as an ideal gas. The compression and expansions during the star's change in size will all be adiabatic processes. We'll first need to show some thermodynamic relations."

Max then recalls from his thermodynamic class that when an ideal gas undergoes an adiabatic process, no heat leaves or enters the closed system. During such a process, the pressure and volume of the ideal gas are related by the following relation:

$$PV^{\gamma} = \text{constant}$$

where γ is the specific heat capacity ratio.

(f) Starting with the ideal gas equation, show that the following relations hold for adiabatic processes. [4]

$$\frac{P_1}{P_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \quad \text{and} \quad \frac{P}{\rho T} = \text{constant.}$$

Hint. *It is also useful to remember that the density is* $\rho = m/V$.

Solution:

Since the density is related to the volume by $V = m/\rho$, it means the adiabatic relation from the hint becomes,

$$P\left(\frac{m}{\rho}\right)^{\gamma} = \text{constant}$$

Moreover, as the mass in each layer is constant, we can ignore it and get the relation between pressure and density,

$$\frac{P_1}{P_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

The next relation to use is the Ideal Gas equation,

$$\frac{PV}{T} = nR = \frac{Pm}{\rho T} = \text{constant}$$

Again, the mass is constant and this gives us the last piece needed,

$$\frac{P}{\rho T} = \text{constant}$$

and,

$$\frac{P_1}{P_2} = \frac{\rho_1 T_1}{\rho_2 T_2}$$

"Ok so we have these thermodynamic relations. It's time to introduce the final piece that is Kramer's law," Annie mentions.

"Kramer's law basically relates the opacity of the gas with its density and temperature."

$$\kappa \propto \frac{\rho}{T^{3.5}}.$$

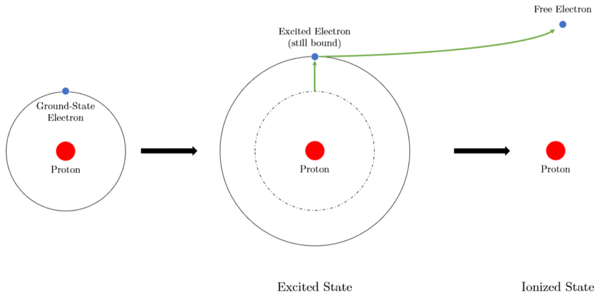

Figure 21: Ionisation of an atom.

"The larger the value of κ, the more opaque the gas." Max continues to work through the math (again) in his head but seems to keep stumbling onto the same contradiction.

(g) Show that the opacity of the gas actually decreases when the star shrinks and the stellar material compresses. It is given that $\gamma = 5/3$. [2]

Solution:

We'll need to relate the opacity before and after compressions. As both the density and temperature of the has changes after compression, we'll need to find a relation. We get those from the two relations we derived earlier.

$$\frac{P_1}{P_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$
$$\frac{P}{\rho T} = \text{constant}$$

We can then combine the two relations,

$$\left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = \frac{\rho_1 T_1}{\rho_2 T_2} \Rightarrow \frac{T_1}{T_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma-1}$$

Now we can introduce Krammer's Law. Denoting parameters with subscript 2 as the parameters of the gas **after** compression,

$$\frac{\kappa_2}{\kappa_1} = \frac{\rho_2}{\rho_1}\left(\frac{T_1}{T_2}\right)^{3.5} = \frac{\rho_2}{\rho_1}\left(\frac{\rho_1}{\rho_2}\right)^{3.5(\gamma-1)} = \left(\frac{\rho_1}{\rho_2}\right)^{4/3}$$

$\rho_2 > \rho_1$ implies $\kappa_1 > \kappa_2$. Therefore, the gas gets less opaque with compression.

κ-mechanism

"Wait, but the math says the star will get more transparent with compression?" Max says.

"Haha, surprise? Actually you're right, it's just that we're missing a crucial piece. It's also the reason why variable stars are not common," Annie mentions.

"Oh ok, you have a point."

"Don't worry, we got bamboozled by the prof also when we were studying this," Fleming reassures, "the stellar material is actually a plasma. However, there are region somewhere in between the core and the 'surface' where the gas is hot enough to get *excited*, but not hot enough to fully ionise and lose that electron. These are the *partial-ionisation* zones in stars."

"In these regions where the stellar material is at the edge between a gas and a plasma, the temperature rise is diminished. A large portion of the atoms are excited but not yet ionised. This is where the magic happens."

(h) Suggest a reason as to why the temperature does not rise as sharply in these partial-ionisation regions within stars. [2]

Solution:

The extra energy in the environment goes towards ionizing the matter and casing the excited atoms to lose the electrons still bound. Hence, not as much energy goes into increasing the speed of the particles and by proxy the temperature of the matter as a whole.

"Ok, I'm starting to see the link as to why this region is crucial for variable stars." Henry mentions after hearing Annie's explanation on why the temperature doesn't increase as much.

(j) Suggest how the presence of this region in the star helps boost the increase in opacity of the stellar material in this region when it gets compressed. [2]

Solution:

By having an increase in density but without a similar change in temperature, the net effect becomes an increase in κ, hence an overall increase in opacity by Krammer's Law.

"The coolest thing about this is that because these ionisation regions occur at specific temperatures, their location from the centre of the star dictates how strongly the star get change in size. This effectively sets the kind of stars that can possibly pulsate," Annie mentions. "If the star is too cool, these regions are found too deep inside the star and other convective effects dampen these oscillations. If the star is too hot, these regions are found further out and that is bad too. I'm sure you can figure out why."

(k) Suggest a plausible reason that explains why stars that are too hot and have these partial ionisation zones further from the core make it hard for oscillations to occur. [2]

Solution:

These partial-ionization zones occur at a specific temperature. In hotter stars, these regions are thus found closer to the surface of the star. By having these ionisation zones further from the core, it's located in the parts of the star that are less dense. This means less matter is driving the oscillations.

The quartet continues to discuss when suddenly one of their alarms goes off.

"Astronomical twilight has ended! The show's starting. I'm picking the northern sky!" Fleming exclaims as she walks towards her chair and re-positions it to face Ursa Minor.

"I'm picking the south-eastern sky," Annie follows.

"Wait will the meteors appear more often in those parts of the sky?" Max asks.

"Eh... it's complicated, but that's a story for another time. Now it's meteor gazing time," Annie says as she lays down on mat and looks up at the heavens.

Max and Henry go to sit in their chairs and start the night observation with a cup of coffee in this winter wonderland. As they looks towards the heavens, they're greeted by the brilliance of a hundred stars.

It’s a perfect night.