A Journey to Mars [18.5 marks]

One New Year's Day, a spaceship was launched from Venus to Mars on an elliptical orbit. The orbital trajectory is such that:

- When the spacecraft reaches aphelion, it will cross the orbit of Mars at a tangent
- When the spacecraft was launched (at perihelion), it emerged from the orbit of Venus at a tangent
- The spacecraft's departure was timed such that when the spacecraft reaches aphelion, Mars will be at the destination
- No course corrections were necessary.

Assume that all planets have circular orbits.
Coincidentally, Venus is at its greatest eastern elongation on this special day.

a) Sketch a diagram of the orbits of Venus, Earth, Mars and the spacecraft. Your diagram should include the initial and final position of Mars. [4 marks]

b) On your diagram, extend a line from the Sun to the initial position of Venus & the spacecraft.Label this line A. Extend another line from the Sun to the final position of Mars & the spacecraft. Label this line B. [0.5 marks]

   What is the interior angle between line A and line B? [1 mark]

c) What is the semi-major axis and eccentricity of this spacecraft's orbit? [2 marks]

d) Suppose that the spacecraft has a mass of 5 tons (without fuel). Furthermore, its rockets have an exhaust velocity of 4200 m/s. If the spacecraft uses up all of its fuel at launch to reach this orbit (requiring a Δv of 7.3 km/s), how much fuel is required to accomplish this maneuver? [2 marks]

   Hint: For the purposes of this question, you can ignore the escape velocity of Venus. Equivalently, you may treat Venus as massless. If it helps, you may also assume the spaceship is filled with spherical cows that walk on frictionless surfaces.

e) How long does the spacecraft take to complete its journey? [2 marks]

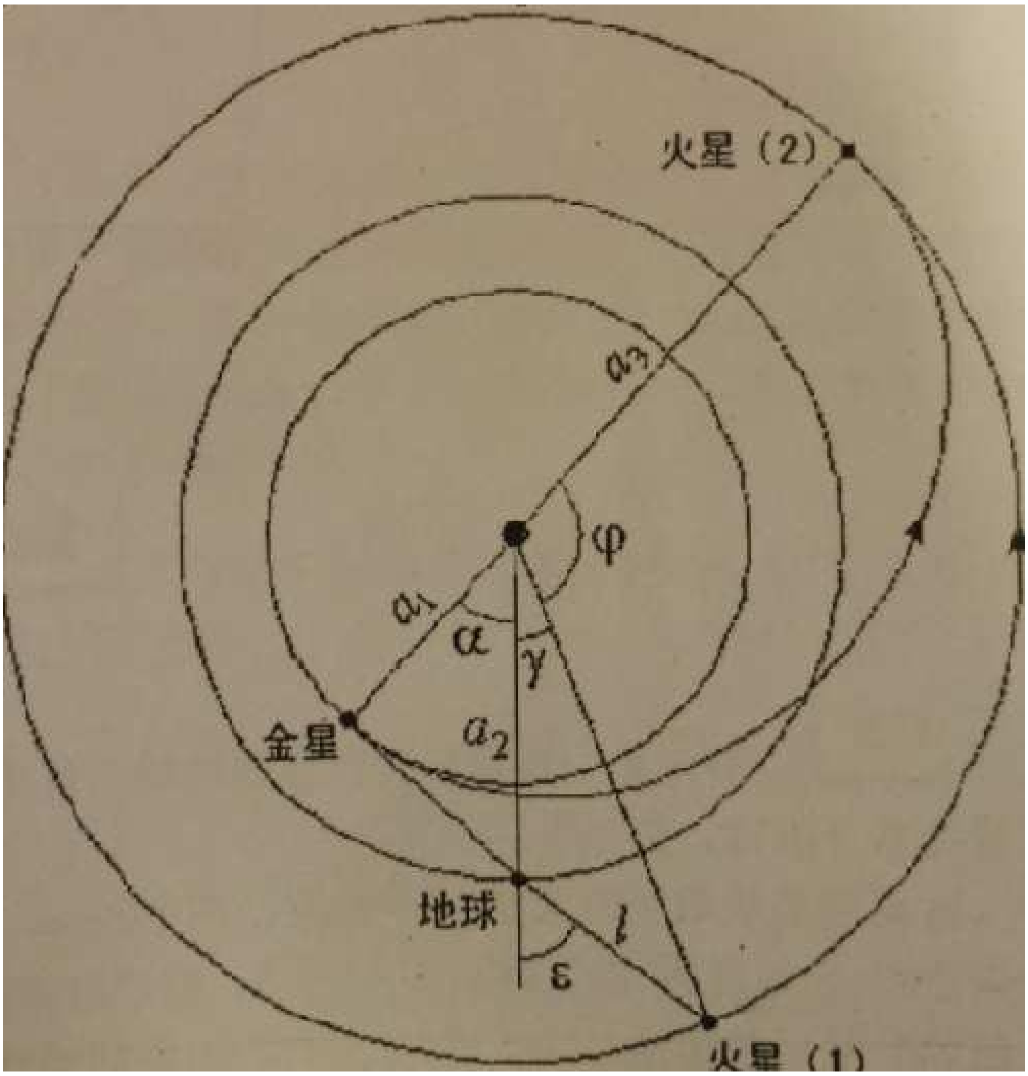
f) In this time frame, Mars has moved by φ degrees along its orbit. Find φ [1 mark]

g) Suppose that on this New Year's Day, the Sun had a RA of $18^h\ 40^m$ from Earth's perspective. Hence, what was the RA of Mars on this day? [5 marks]

h) Name a bright star that would have a RA near Mars. [1 mark]

Solution

a)



Known:

a_{venus}= 0.723AU

a_{earth}= 1AU

a_{mars}= 1.524AU

b) Refer to above diagram. It should be clear that the interior angle is 180 degrees. Mathematically, as Line A and B connects the periapsis of the spacecraft to the apoapsis of the spacecraft, the two lines have to merge together to form a single straight line.

c) We know that $r_{perihelion} = a_{Venus}$ and$r_{aphelion} = a_{Mars}$ The major axis is then $a_{Venus} + a_{Venus}$, and thus the semi-major axis is $\frac{1}{2}(a_{Venus} + a_{Venus}) = 1.68 \times 10^{11}m = 1.12\ AU$

$$\epsilon = \frac{1.68 \times 10^{11}m - 1.082 \times 10^{11}m}{1.68 \times 10^{11}m} = 0.356$$

d)

$$7300 = 4200 \ln \frac{5 + m_{fuel}}{5}$$
$$\frac{5 + m_{fuel}}{5} = e^{\frac{73}{42}}$$

$$m_{fuel} = 5e^{\frac{73}{42}} - 5 = 23.43\ tonnes$$

e) Since the Earth takes 1 year to complete its 1AU orbit, we can quickly come up with a nice form of Kepler's Third Law (within the solar system)

$$T^2 = a^3$$

Where T is in years and a is in AU.

Since the spacecraft only travels along half of its orbit, the time taken t is given by

$$t = \frac{1}{2}(1.12)^{1.5} = 0.593\ years$$

f) We know that Mars takes 686.97 days to complete its orbit, or 1.88 years. Thus:

$$\varphi = 360 \times \frac{0.593}{1.88} = 113.43°$$

g) Referring to the diagram above:

$$\alpha = \arccos \frac{a_1}{a_2} = 43.7°$$

The position of Sun , position of Earth and current position of Mars form a triangle.

$$\gamma = 180^o - \alpha - \phi = 22.4^o$$

So $l = \sqrt{a_2^2 + a_3^2 - 2a_2 a_3 cos\gamma} = 0.710 AU$

$$\epsilon = arcsin[\frac{a_3 * sin\gamma}{l}] = 54.9^o$$

We know that the right ascension of Sun is about 18 h 40m. From the calculation above, the right ascension of Mars is about 6.67 h+ e [which is 54.9/15 h]. This gives us a RA of 10.33 h.

h) This is close to the RA of Regulus.

Hubble's Law for Messier Spirals [20 marks]

Does Hubble's Law hold up even for relatively nearby galaxies? In this question, we will attempt to determine the validity of Hubble's Law within the Messier Catalogue. For ease and consistency, we will restrict ourselves to spiral galaxies in the Messier Catalogue (n=25). Key data is given below.

Object	Distance (Mpc)	Redshift (km/s)	M
M31	0.89	-300	-21.34
M33	0.92	-180	-19.12
M51	11.34	462	-21.87
M58	20.85	1518	-21.90
M61	18.40	1566	-21.62
M63	11.34	504	-21.67
M64	5.83	408	-20.33
M65	10.06	807	-20.71
M66	10.73	729	-21.25
M74	10.73	657	-20.75
M77	18.40	1137	-22.42
M81	3.68	-33	-20.93
M83	4.60	516	-20.71
M88	18.40	2283	-21.72
M90	18.40	-234	-21.82
M94	4.45	309	-20.04
M96	11.65	897	-21.13
M98	18.40	-141	-21.22
M99	18.40	2409	-21.42
M100	18.40	1572	-22.02
M101	8.28	240	-21.69
M104	15.33	1092	-22.93
M106	7.67	447	-21.02
M108	13.80	699	-20.70
M109	16.86	1050	-21.33

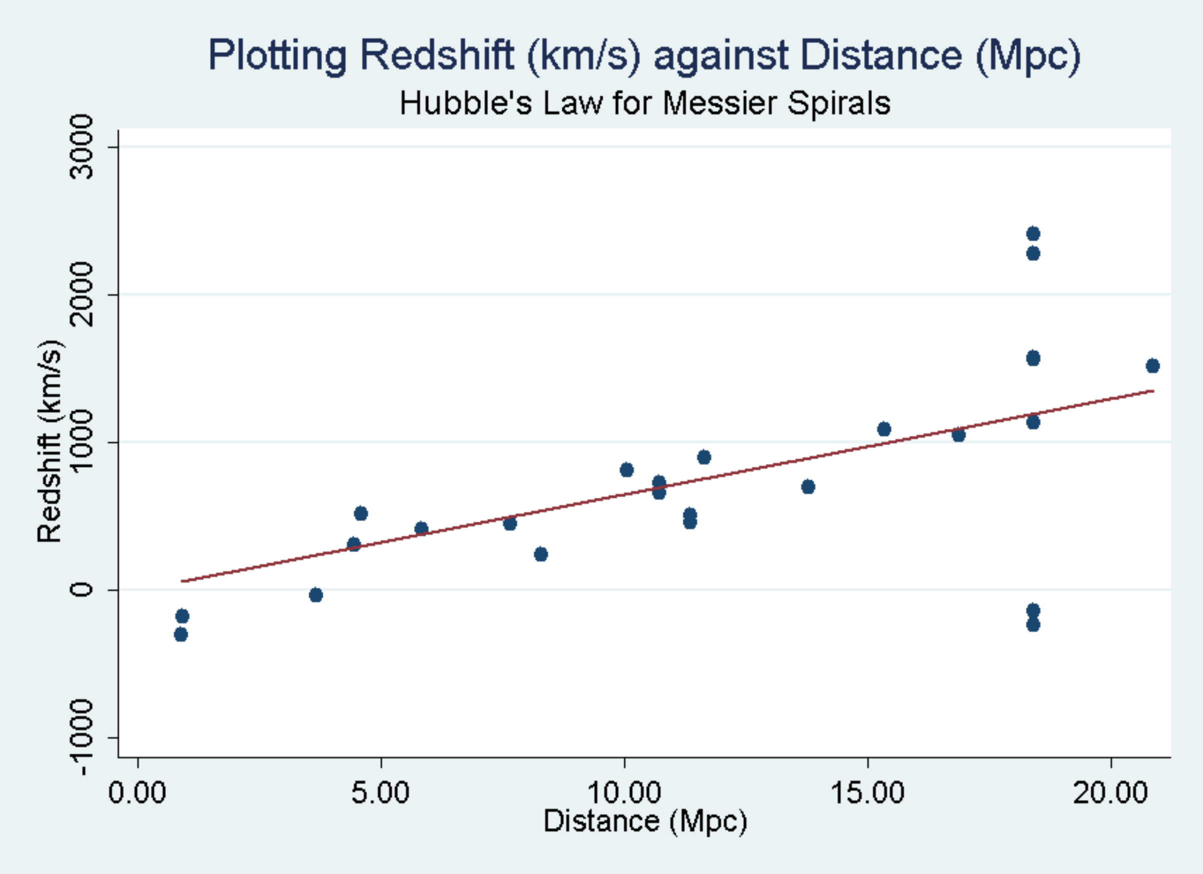
a) From the data above, determine the apparent magnitude of M33. **[1 mark]**

Distance modulus = $5\log\frac{0.92\times10^{6}}{10} = 24.8$

Thus the apparent magnitude is -19.1+24.8 = 5.7

b) Plot a graph of redshift against distance for the 25 galaxies listed above. Plot a best-fit line. **[4 marks]**

Note: The best-fit line should be drawn such that the intercept should be 0. Can you see why?



c) From the data above, estimate Hubble's Constant. **[2 marks]**

Let the variable 'Mpc' denote the distance in Mpc. Then:

```
. regress v Mpc, noconstant

      Source |       SS       df       MS              Number of obs =      25
-------------+------------------------------           F(  1,    24) =   60.72
       Model |  18770188.5     1  18770188.5           Prob > F      =  0.0000
    Residual |  7419523.51    24  309146.813           R-squared     =  0.7167
-------------+------------------------------           Adj R-squared =  0.7049
       Total |    26189712    25  1047588.48           Root MSE      =  556.01

------------------------------------------------------------------------------
           v |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         Mpc |   64.90868   8.330112     7.79   0.000     47.71617    82.10119
------------------------------------------------------------------------------
```

For the non-statistically inclined, STATA is essentially saying that Hubble's Law is estimated to be the following equation: $v = 64.9Mpc$

Thus, H_0= 64.9km/s per Mpc

d) When you compare your value in c against the value of the Hubble Constant as given in your Formula book, you find that your estimate is slightly different than what you expect. Suggest a reason why this is the case **[2 marks]**

The gravitational attraction of the Milky Way/ Local Group is significant at small distances, decreasing the apparent value of the Hubble Constant.

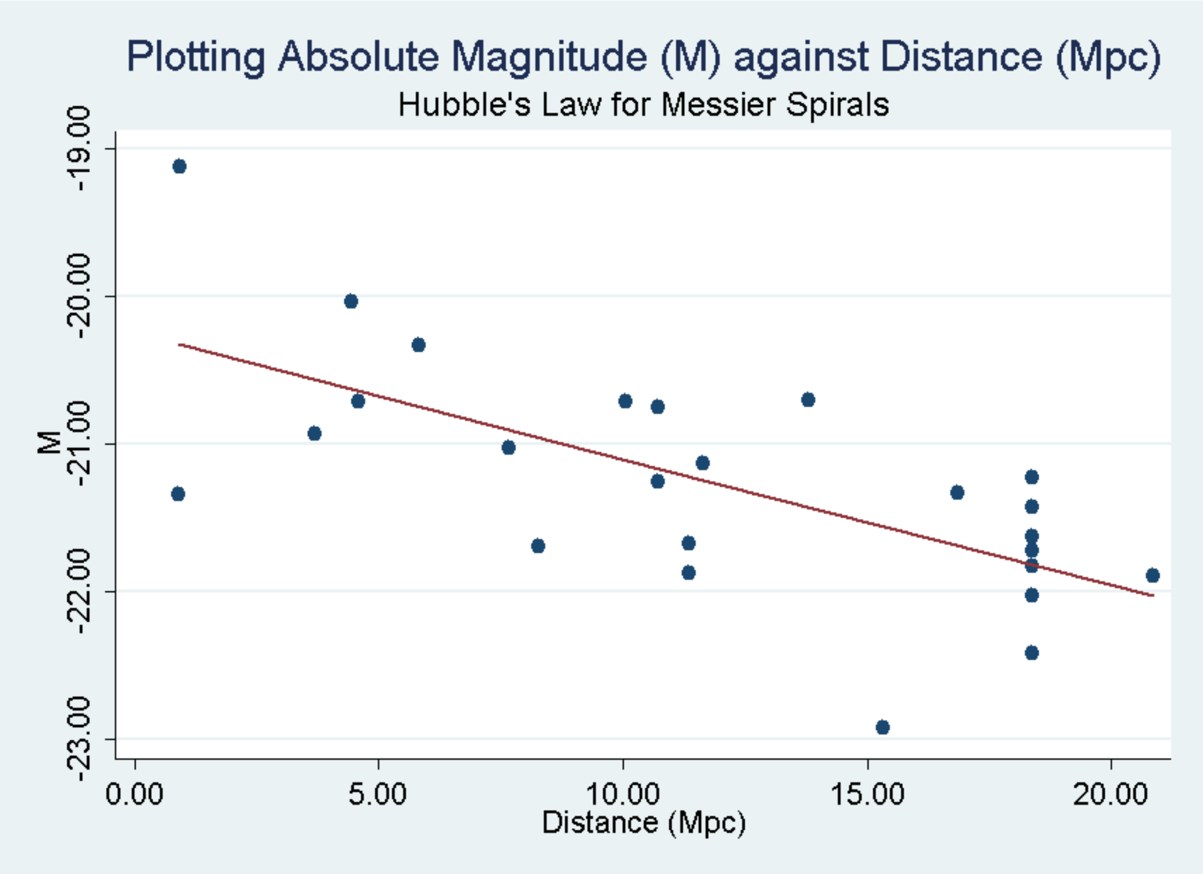
e) You should have noticed a prominent clump of galaxies at 18.4 Mpc. This clump is due to the presence of multiple members of the Virgo Cluster. Due to the gravitational attraction of the Virgo Cluster, these galaxies have significant orbital velocities (in addition to the Hubble Flow).

Some of the galaxies in the Virgo Cluster appear to be large outliers on your graph. Suppose that the largest outliers are because the orbital velocity of the said galaxy is directly towards us (or away from us).

Hence or otherwise, estimate the orbital velocity within the Virgo Cluster. **[2 marks]**

At 18.4 Mpc, the predicted redshift (from the best-fit line is v=1194 km/s. M90 has the greatest residual (aka it is the greatest outlier). Taking the difference gives an absolute value of 1428 km/s. [M99 is also accepted, which gives a value of 1215 km/s]

f) Plot a graph of absolute magnitude, M against distance for the 25 galaxies listed above. Plot a best fit line. **[4 marks]**



g) If there was a spiral galaxy at 1000 Mpc that followed this best-fit line, what would be its absolute magnitude? **[2 marks]**

```
. regress M Mpc

      Source |       SS       df       MS              Number of obs =      25
-------------+------------------------------           F(  1,    23) =   18.59
       Model |  6.64074937     1  6.64074937           Prob > F      =  0.0003
    Residual |  8.21776013    23  .357293919           R-squared     =  0.4469
-------------+------------------------------           Adj R-squared =  0.4229
       Total |  14.8585095    24  .619104563           Root MSE      =  .59774

------------------------------------------------------------------------------
           M |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         Mpc |   -.085497   .0198315    -4.31   0.000    -.1265215   -.0444724
       _cons |  -20.24983   .2647381   -76.49   0.000    -20.79748   -19.70218
------------------------------------------------------------------------------
```

STATA says that the equation of the best fit line is essentially:

$$M = -20.25 - 0.085Mpc$$

So when Mpc=1000 (aka the galaxy is 3.26 billion light years away),

$$M = -20.25 - 85 = -105.25$$

This is obviously crazy

h) Hence or otherwise, state a significant bias in the Messier Catalogue. In other words, in what way are the distant spiral galaxies in the Messier Catalogue different from spiral galaxies at similar distances? **[1 mark]** Briefly suggest how this bias appears **[2 marks]**

They are brighter/more luminous than the average spiral galaxy at that distance. This is an example of Malmquist bias (not needed in answer). As objects rapidly appear dimmer as distance increases, only the brighter spiral galaxies can be seen from Earth. Hence, the galaxies in the Messier catalogue will tend to be brighter than the average galaxy at that distance.

NB: any other reasonable selection biases were accepted (e.g. Charles Messier being unlikely to mistake faint galaxies as comets)

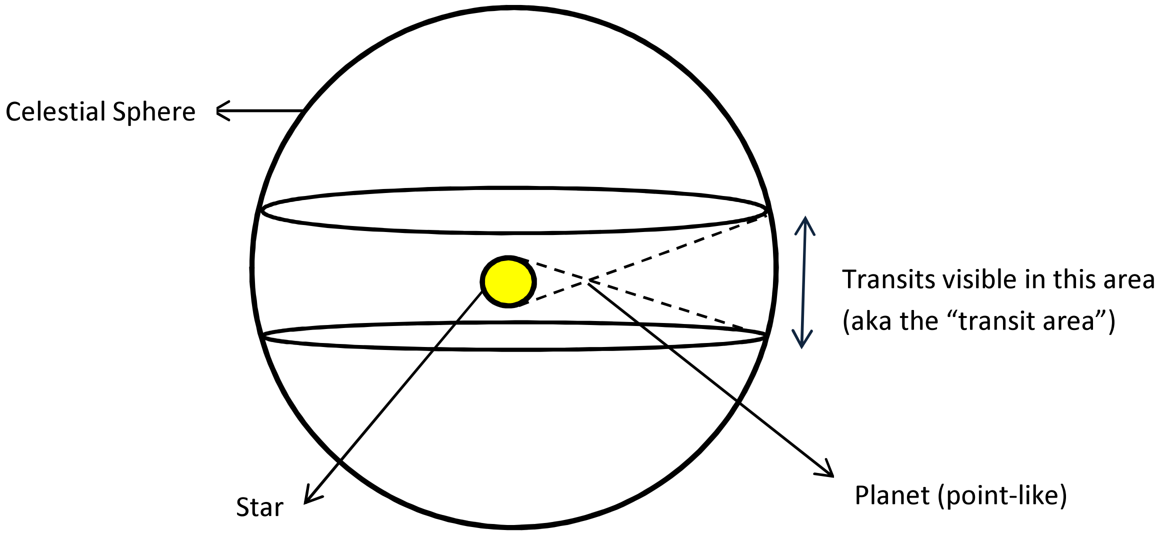
Searching for Earths [21 marks]

Suppose your teams is about to launch a search for habitable exoplanets. Which stars should you focus on? This question attempts to explore two different viewpoints on this issue.

Transiting exoplanets [6 marks]

Suppose a planet is orbiting its host star in a circular orbit. As it does so, it sweeps out an area on the celestial sphere where some observers will be able to witness transits. See the sketch below. (Adapted from Borucki and Summers, 1984)

Let us define this area as the "transit area". By comparing the "transit area" to the "surface area" of the celestial sphere, we are able to estimate the probability that we can detect a transiting exoplanet.



a) Let the semi-major axis of the planet's orbit be a, and the radius of the star be r_{star}. From the perspective of the planet, what is θ, the angular size of the star? Note that θ is small. **[1 mark]**

Since θ is small

$$\tan\frac{\theta}{2} = \frac{r_{star}}{a} \rightarrow \tan\theta = \frac{2r_{star}}{a}$$

$$\theta \approx \tan\theta = \frac{2r_{star}}{a}$$

b) Hence or otherwise, show that the probability the exoplanet transits the face of its host star from our perspective is given by the formula below **[5 marks]**

$$P_{transit} \approx \frac{r_{star}}{a}$$

Hint: You may treat the distance between the planet and the celestial sphere as D. Note that D is **nearly infinite**, (aka a << D). Remember to explain your workings.

The width of the strip swept out, s=Dθ

The strip length is approximately the entire circumference of the circle, which is 2π(D+a)

Hence the area of the strip is 2π(D+a) × Dθ

The "surface area" of the celestial sphere is $4\pi(D+a)^2$. Hence:

$$P_{transit} = \frac{2\pi(D+a)D\theta}{4\pi(D+a)^2} = \frac{D\theta}{2(D+a)}$$

Since D >> a and $\theta = \frac{2r_{star}}{a}$

$$P_{transit} \approx \frac{D}{2D}\left(\frac{2r_{star}}{a}\right) = \frac{r_{star}}{a}$$

Habitability [9 marks]

From the formula above, it seems like exoplanets are easier to detect around small stars. On its own, that suggests that we should focus on red dwarfs. But, even if we find an exoplanet around a red dwarf, is that planet necessarily conducive to life?

In order to answer that, we'll attempt to explore the properties of the Habitable Zone

c) Suppose the planet reflects a fraction (A) of all incoming radiation. In other words, the planet has an albedo of A. Furthermore, assume the planet is not a perfect blackbody: rather each unit area of the planet follows a modified form of the Stefan-Boltzmann law, as given below.

$$L = \varepsilon\sigma T^4$$

Where ε (its emissivity) is a constant that ranges from 0 to 1. Given that the star has a luminosity of L, solve for the equilibrium temperature of the planet, and show that it is independent of its size. **[3 marks]**

Suppose the planet has a radius of r_p
The energy absorbed by the planet is:

$$E_{absorbed} = L \times \frac{\pi {r_p}^2}{4\pi a^2} \times (1-A)$$

$$E_{emitted} = \varepsilon\sigma T^4 \times 4\pi {r_p}^2$$

$$E_{absorbed} = E_{emitted} \rightarrow \varepsilon\sigma T^4 \times 4\pi {r_p}^2 = L \times \frac{\pi {r_p}^2}{4\pi a^2} \times (1 - A)$$

$$T^4 = (1 - A)\frac{L}{16\pi\varepsilon\sigma a^2}$$

$$T = \sqrt[4]{\frac{L(1 - A)}{16\pi\varepsilon\sigma a^2}} = \frac{1}{2\sqrt{a}}\sqrt[4]{\frac{L(1 - A)}{\pi\varepsilon\sigma}} \ (Q.E.D)$$

d) Suppose a planet is habitable if its equilibrium surface temperature is exactly T_h. Hence or otherwise, derive a formula for the radius of the Habitable Zone, a_h. **[2 marks]**

$$T_h = \frac{1}{2\sqrt{a_h}}\sqrt[4]{\frac{L(1 - A)}{\pi\varepsilon\sigma}}$$

$$\sqrt{a_h} = \frac{1}{2T_h}\sqrt[4]{\frac{L(1 - A)}{\pi\varepsilon\sigma}}$$

$$a_h = \frac{1}{4{T_h}^2}\sqrt{\frac{L(1 - A)}{\pi\varepsilon\sigma}}$$

e) Suppose we relax this assumption, and claim that planets with an equilibrium surface temperature within a constant ΔT of T_h are habitable. This causes the Habitable Zone to have a width, W, rather than the single value in part e. Find an expression for the width of the habitable zone. **[2 marks]**

First, we need to find the inner and outer boundaries of the Habitable Zone

$$a_{inner} = \frac{1}{4(T_h + \Delta T)^2}\sqrt{\frac{L(1 - A)}{\pi\varepsilon\sigma}}$$

$$a_{outer} = \frac{1}{4(T_h - \Delta T)^2}\sqrt{\frac{L(1 - A)}{\pi\varepsilon\sigma}}$$

$$W = a_{outer} - a_{inner}$$

$$W = \frac{1}{4(T_h - \Delta T)^2}\sqrt{\frac{L(1 - A)}{\pi\varepsilon\sigma}} - \frac{1}{4(T_h + \Delta T)^2}\sqrt{\frac{L(1 - A)}{\pi\varepsilon\sigma}}$$

$$W = \sqrt{\frac{L(1 - A)}{\pi\varepsilon\sigma}}[\frac{1}{4(T_h - \Delta T)^2} - \frac{1}{4(T_h + \Delta T)^2}]$$

f) Find an expression that describes how W changes with luminosity L. Show that more luminous stars have a wider habitable zone than less luminous stars **[2 marks]**

Since everything else is a constant, we can differentiate this directly.

$$\frac{dW}{dL} = \frac{1}{2\sqrt{L}}\sqrt{\frac{1-A}{\pi\varepsilon\sigma}}\left[\frac{1}{4(T_h - \Delta T)^2} - \frac{1}{4(T_h + \Delta T)^2}\right]$$

All the terms in this expression are positive, and hence more luminous stars have wider Habitable Zones

NB: Other methods of arguing this were accepted (e.g. $W \propto \sqrt{L}$)

Analysis [6 marks]

So far, there are two opposing factors to consider. While it is easier to find exoplanets around red dwarfs, more luminous stars have a wider Habitable Zone, and thus are more likely to host habitable planets.

g) Based on your prior knowledge, briefly argue whether your search for habitable planets should focus on more luminous stars or less luminous red dwarfs. Your answer should be well reasoned and compare two **distinct** advantages/disadvantages between these two types of stars. In other words, you should not reuse the two opposing factors that have already been mentioned in this question. **[6 marks]**

Reasonable answers were accepted

If you wanted to argue that more luminous stars were better (pick 2, not exhaustive)

- Habitable planets near red dwarfs would be tidally locked, which may pose difficulties for the biosphere (e.g. possibility that all water freezes into ice on the far side, especially if the planet does not have a thick atmosphere/oceans to efficiently transport heat).
- Close proximity also leads to a higher probability of experiencing damaging solar flares.
- Wider separation of planet and star would allow easier study of such exoplanets, especially for nearby luminous stars (direct imaging becomes a possibility)

If you wanted to argue that less luminous red dwarfs were better (pick 2, not exhaustive)

- Shorter lifespan of more luminous stars means there is less time for life to develop.
- Red dwarfs emit comparatively less radiation in UV due to their cooler temperatures, reducing the probability of damaging mutations to living organisms.
- Exoplanets around red dwarfs would create larger "wobbles" around their host stars, and would thus be easier to detect through the radial velocity technique.

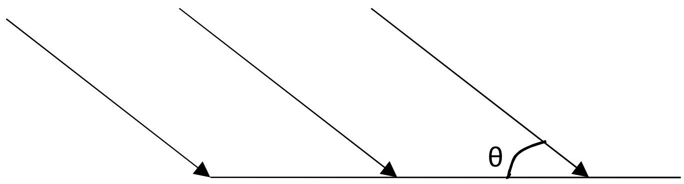
NB: If you understood the reasoning behind the calculations, you should realise that <u>planets in the Habitable Zone experience exactly the same amount of radiation</u>! Thus, as long as you are in the Habitable Zone, the type of star does not further influence the <u>amount</u> of radiation a planet receives. Therefore this was not accepted as a valid reason.

Seasons in the Sun [23.5 marks]

How much hotter are the tropics compared to other places in winter? Here, we'll take a stab at this question.

Inclines

While the sun pumps out 1370 W of solar radiation per square meter [aka the solar constant], that does not mean every square meter of Earth receives that amount of energy every second!



a) Consider the setup above. Parallel rays of sunlight strike a surface (area of 1 m^2) at an angle. Write down an expression for the energy absorbed per second by 1 m^2 of the exposed surface, in terms of θ. **[0.5 marks]**

Resolve the components: you'll find that the perpendicular component that strikes the surface is:

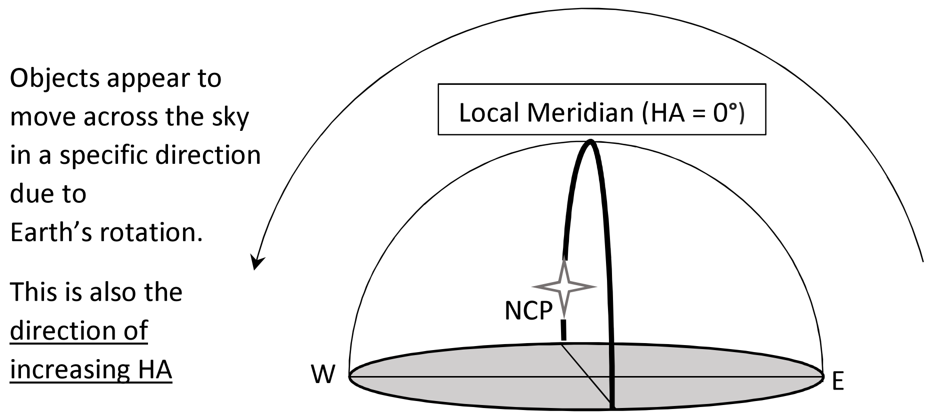
1370 sin θ

Hour Angles

In astronomy, the concept of the **Hour Angle [HA]** is often used. In general, the HA is the angular distance of a point measured westward from the Local Meridian (the upper arc connecting the zenith to the celestial poles).

The HA can be expressed in terms of angles (degrees/radians/arcseconds etc), or units of time like hours/minutes/seconds. If we adopt the latter convention, the HA represents the amount of time since the object last crossed the meridian, or the time before its next transit. Indeed, hour angles are often expressed in this form. However, for ease of computation, we will stick to **angles** for this question.

A diagram for an observer in the Northern Hemisphere is shown below. For clarity, the South Celestial Pole has been omitted.



Determining the HA for any object at a specific time can be computationally demanding. However, calculations for the Sun are greatly simplified if we use solar time, as we can use more natural units. For example, the Sun has an HA of 0° at solar noon (as it must be on the Local Meridian then).

For instance, what is the HA of the Sun 1 hour after solar noon? As we know that the Sun roughly makes a complete 360° loop in 24 hours, the answer is simply the apparent angular velocity of the Sun [aka $\frac{360°}{24} = 15°$]. Note that we'll confine the HA between $\underline{-180° \text{ to } 180°}$.

b) Suppose we are on Mars, which has a day 40 minutes longer than ours. What is the HA of the Sun 2 hours before solar noon? Leave your answer in degrees. **[1 marks]**

$$0 - \frac{360°}{24.667} \times 2 \approx -29.19°$$

Rising and setting

With the help of spherical geometry, we can calculate when objects rise and set. For instance, when the sun is on the horizon, its HA [henceforth known as $HA_{horizon}$] is given by

$$\cos HA_{horizon} = -\tan \delta_{sun} \tan l$$

Where δ_{sun} represents the declination of the Sun for that day and l represents the geographical latitude of the observer. l ranges from $-90° \leq l \leq 90°$, with negative values denoting that the observer is south of the Equator. Due to the Earth's axial tilt, the declination of the sun ranges from $-23.5° \leq \delta_{sun} \leq 23.5°$

c) Will the above equation always yield real solutions for $HA_{horizon}$, given the ranges specified for l and δ_{sun}? **[1 mark]** Note: A mathematical proof is overkill, intuition and/or examples will suffice

No. The formula does not always work beyond the Arctic/Antarctic Circle, where the Sun does not rise/set during certain days of the year.

d) Suppose $l = 40°$ and $\delta_{sun} = -23.5°$. What is the HA of the sun at sunrise and sunset? **[0.5 marks]** Hence, what is the length of the day given these conditions? Ignore atmospheric effects like refraction, twilight and scattering etc. **[2 marks]**

$$|HA_{horizon}| = \arccos(-\tan -23.5° \tan 40°) \approx 68.60°$$

The time between sunrise and local noon (or local noon to sunset) is then $68.6° \div \frac{15°}{hour} = 4.57\ hours$

The length of day is then simply 4.57 hours × 2, or **9.15** hours.

e) For the same location and date, determine how long Betelgeuse [DE: +07° 24′ 25.426″] is above the horizon **[2.5 marks]**

Different context, same approach. Simply plug in the declination of Betelgeuse into the equation!

$$|HA_{horizon}| \approx \arccos(-\tan 7.4° \tan 40°) \approx 96.26°$$

$$Time\ above\ horizon = 96.26° \times 2 \div \frac{15°}{hour} = 12.83\ hours$$

f) A student points out that the amount of time Betelgeuse is above the horizon can be very different from the amount of time Betelgeuse is *actually visible.* For all we know, Betelgeuse might just rise around the same time as sunrise. Do you think this is a significant concern? Explain **[2 marks]**

No. [1 mark] During the winter solstice, the Sun and Orion are approximately on opposite ends of the sky. [1 mark] Recall that Orion is a winter constellation! Hence, Betelgeuse will rise after sunset and set before sunrise

Altitude

Define the angle the sun makes with the horizon [aka its altitude] at any point in time as θ. From spherical geometry, we can find θ by the following equation:

$$\sin\theta = \cos HA \cos\delta_{sun} \cos l + \sin\delta_{sun} \sin l$$

You may want to verify that if $l = 40°$, the formula above gives the correct values for the maximum altitude of the Sun during the summer and winter solstices. This is left as an optional exercise to the reader.

Two roads diverged… [10 marks]

Now we are ready to actually attack the question.

g) How much energy is absorbed by 1 m^2 of Earth in 1 day for the following surfaces?

1. The surface of Earth at the Equator during either Solstice
   (Can you see why the choice of Solstice doesn't matter?)
2. The surface of Earth at 40 N during the Winter Solstice

There are two ways to go about this. Either method is perfectly acceptable and will give the same amount of marks. However, as both methods are based on the same logic, you might find it helpful to read both methods (and their hints) before making your choice.

Note: We will NOT double count credit if you simultaneously submit an answer from both methods. In fact, we will only grade the more incorrect answer!

Method 1

Construct and evaluate a suitable definite integral for each case **[5 marks each]**.

For starters, you should be thinking of integrating a function relating the altitude of the Sun(θ) to the energy absorbed by the surface. This brings to mind your answer in part a). HOWEVER, working with θ means you will have to work with a line integral, which is NOT FUN.

To avoid this, express your answer in part a) with HA instead. Substitute all functions involving θ with functions using HA, **then** integrate. <u>You will find the previous formulae useful in this regard.</u>

What are the integration limits then? As we are working with a physical problem, there's a readily understandable answer: you start from the value of the Sun's HA at sunrise, and end with the value of the Sun's HA at sunset. By integrating from sunrise to sunset, you'll end up with the total amount of energy received from the sun in 1 day.

At the end of it all, you should get an integral like this

$$\int_{HA_{sunrise}}^{HA_{sunset}} C \times A(HA)\, d(HA)$$

Where:

- A(HA) is your answer in part a), expressed in terms of HA
- C is a constant that pops out after converting everything into radians. Remember that you should always evaluate your integrals using radians!
- HA_{sunset} and $HA_{sunrise}$ are constants to be found

Hint 1: A(HA) may not be the prettiest function in the world, but it should only involve HA and constants. Evaluate the constants, and A(HA) will be simplified greatly.

Hint 2: You might wonder: what is the solar flux per radian? Well, when the Sun has moved by 1 radian, $\frac{1}{2\pi}\ days \approx 13751\ s$ have elapsed...

Method 2

For each scenario, plot the energy received by the surface at different points in time. Hence or otherwise, estimate the total amount of energy received by each surface across a day. **[5 marks each]**. Full credit will be awarded only if your estimates are within 10% of the correct value.

Hint 3: You can collect your graph paper from the friendly QMs

Hint 4: The straightforward way of calculating & plotting points from sunrise to sunset is tedious. There's a smarter and much better way to do things.

Answer:

Plugging in the altitude formula into the answer in part a, we get:

$$A(HA) = 1370[\cos HA \cos \delta_{sun} \cos l + \sin \delta_{sun} \sin l]$$

Since we need to convert the solar flux in terms of radians (not seconds), $C = 13{,}751$

In order to perform this substitution correctly, you first need to find the limits for the integral. Working in radians for the case of 40 N, you should get:

$$|HA_{horizon}| = \arccos(-\tan -23.5° \tan 40°) \approx 68.60° \approx 1.197\, rad$$

Upon some reflection, observe that **we can simply integrate for half a day** (from sunrise to noon or noon to sunset) and multiply by 2. This saves a lot of work!

Plugging everything in, we get:

$$\int_{HA_{sunrise}}^{HA_{sunset}} C \times A(HA)\, d(HA) \quad = 2\int_{0}^{1.197} (1370 \times 13751)[\cos HA \cos \delta_{sun} \cos l + \sin \delta_{sun} \sin l] dHA$$

This equation looks daunting, but nearly everything is in the form of constants. Simplifying, we get

$$2(1370 \times 13751) \int_{0}^{1.197} [\cos HA\,(\cos -23.5\,°)(\cos 40°) + (\sin -23.5°)(\sin 40\,°)]\; dHA$$

$$\approx 2(1370 \times 13751) \int_{0}^{1.197} 0.7025 \cos HA - 0.2563 \;\; dHA$$

$$= 37{,}677{,}740[0.7025 \sin HA - 0.2563\; HA]_{0}^{1.197} \approx 13.08\; MJ$$

The integral for the Equator is similarly constructed. It is easy to see that $|HA_{horizon}| = \frac{\pi}{2}$. The integral is also easier to deal with. Briefly, you should have:

$$2(1370 \times 13751) \int_{0}^{0.5\pi} [\cos HA \cos \delta_{sun} \cos l + \sin \delta_{sun} \sin l]\; dHA$$

$$= 2(1370 \times 13751) \int_{0}^{0.5\pi} 0.917 \cos HA \;\; dHA$$

$$= 37{,}677{,}740[0.917 \sin HA]_{0}^{0.5\pi} \approx 34.55\; MJ$$

The Answer [4 marks total]

h) Hence, how many times more energy is absorbed at the equator compared to the surface at 40 N during the winter solstice? **[1 mark]**

$$\frac{34.55\; MJ}{13.08\; MJ} = 2.64\; times$$

NB: Accept alternative interpretations of "times", as it is ambiguous.

i) If both surfaces were ideal black bodies, what would be their effective temperature?

Assume that both surfaces maintain a uniform temperature throughout the day. **[2 marks]**

$$86400\sigma {T_{eq}}^{4} = 34.55\; MJ$$

$$T_{eq} \approx 290\; K = 17\; °C$$

$$86400\sigma {T_{40}}^{4} = 13.08\; MJ$$

$$T_{40} \approx 227\; K = -46\; °C$$

j) Even at the equator, the temperature you've calculated seems too cool. Suggest a possible reason why the temperatures you've calculated are lower than expected. **[1 mark]**

- Earth's atmosphere effectively traps heat through the greenhouse effect, making Earth warmer than it would be otherwise

<u>The Milky Way (17 marks)</u>

The Milky Way is a barred spiral galaxy with a supermassive black hole (hypothetically) at its center. If we measure the orbital velocity of every star in Milky Way and plot it on a curve, this is the result:

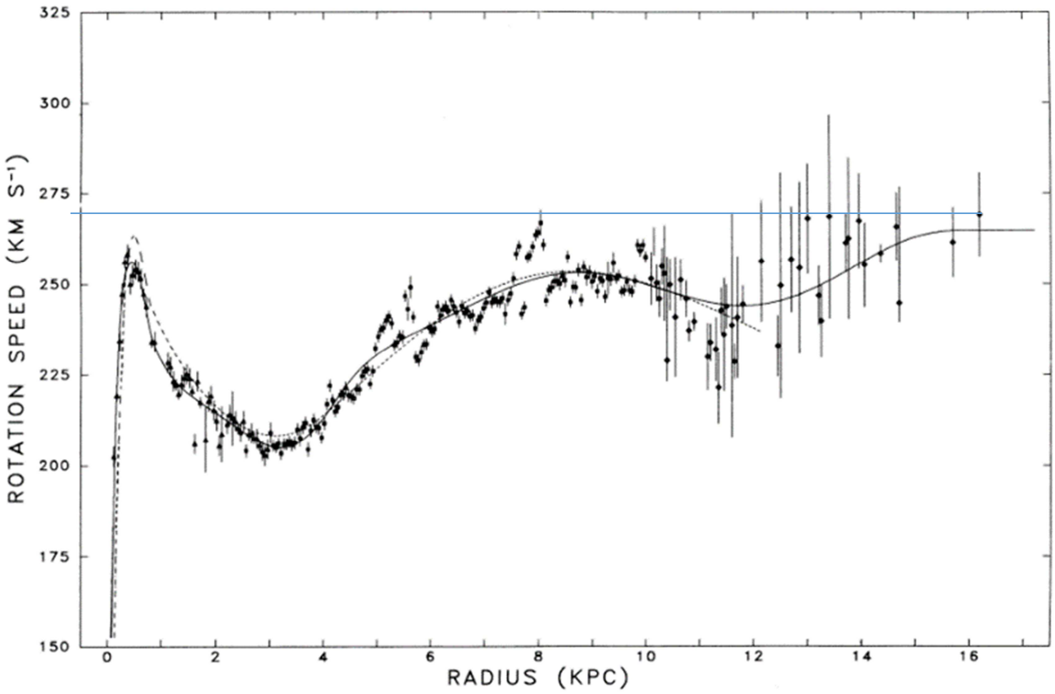


Figure 1: Velocity Curve of Milky Way (from here)

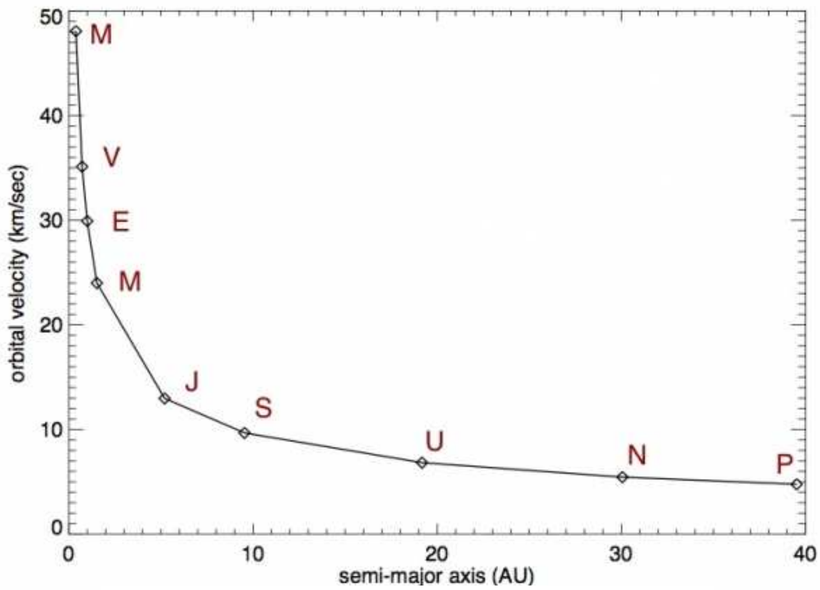


Figure 2: Velocity Curve of Solar System (taken from here)

a) As a first approximation, we may say that the Milky Way and Solar System are rather similar: a very massive object at the center, and everything else revolving around it. However, if we compare the velocity curve of the Milky Way (Figure 1) with our solar system (Figure 2), it looks very different at large distances. What do you think is the difference? Relate your answer with Newton's Law of Gravity. **[3 marks]**

Newton's Law of Gravity relates gravity force with mass, and total mass of the system is responsible for the force. For Solar System, 99.8% of its total mass comes from the Sun itself. As the orbit goes bigger, the number of mass does not increase significantly, making the orbital velocity drops due to distance increase (and mass increase cannot offset it). On the other hand, mass distribution of Milky Way is not as centralized as Solar System. Stars are everywhere, and as star's orbit goes larger the number of other stars inside the orbit will increase quite significantly. Therefore the orbital velocity in Milky Way is quite flat.

b) Using the data given, estimate the approximate total mass of the Milky Way. Ignore the error bars shown on the curve. **[3 marks]**

Assume all starts orbit in a circular orbit. Circular velocity can be defined with

$$V = \sqrt{\frac{GM}{r}}$$

While M is total mass inside the orbit. The formula can be modified to become

$$M = \frac{V^2 r}{G}$$

From the curve, the outermost star has orbital velocity of 270 km/s at distance of 16 kpc

$$M = \frac{(270000\frac{m}{s})^2 \times (16000 \times 206265 \times 1.5E11\ m)}{6.673E-11\frac{m^3}{s^2 kg}} = 5.4 \times 10^{41} kg$$

c) If we assume there are 50 billion Sun-like stars in our galaxy, what is the luminosity and absolute magnitude of our galaxy? For reference, the Sun's luminosity is 3.86×10^{26} W, Sun's mass is $2\text{x}10^{30}$ kg and its absolute magnitude is 4.8. **[4 marks]**

*If Milky Way consists of 50 billion Sun-like stars, then total luminosity of the galaxy is 50 billion * 3.86E26 W = 1.93E37 W.*

We use this total luminosity to find Milky Way's absolute magnitude by comparing it with a single Sun.

$$M_{galaxy} - M_{sun} = -2.5 \log n$$

$$M_{galaxy} - 4.8 = -2.5 \log 50E9$$

$$\underline{M_{galaxy} = 4.8 - 26.75 = -21.95}$$

d) The mass of the supermassive black hole at the center of our galaxy is approximately 8.2×10^{36} kg. Calculate the total mass of the galaxy based on supermassive black hole and mass of stars from point c. Compare your answer with the value you get at point b. **[3 marks]**

Total star mass = $50\text{x}10^{9}$ x $2\text{x}10^{30}$ kg = 10^{41} kg
Black hole mass = $8.2\text{x}10^{36}$ kg
Total galaxy mass = 10^{41} + $8.2\text{x}10^{36}$ = $1.000082\text{x}10^{36}$ kg
The mass calculated in point d) is lesser than in point b)

e) Recall that for main sequence stars like our Sun, the relationship between luminosity and mass is given by:

$$L \propto M^{3.5}$$

Find the mass-to-light ratio for our galaxy (following the assumption from point c for luminosity but mass from point b). Compare your answer with above relationship. What can you conclude? Relate your answer with point d. **[4 marks]**
The relationship holds if both L and M is declared in Sun unit, therefore we need to convert the value first:
Luminosity = $5\text{x}10^{10}$ L_{sun} (since there are 50 billion Sun-like stars)
Mass = $5.4\text{x}10^{41}$kg/$2\text{x}10^{30}$kg = $2.7\text{x}10^{11}$ M_{sun}

For Sun, the mass-to-light ratio must be 1 (because both mass and luminosity are declared in Sun unit), and if we use mass from point d and light from point c, then the mass-to-light ratio will be 1 as well since all values come from Sun.
However, if we take mass from point b, we will get mass-to-light ratio of $2.7\text{x}10^{11}$/$5\text{x}10^{10}$ = 5.4. This value means 5.4 solar mass in galaxy corresponds to 1 solar luminosity, which is quite low. Therefore, the galaxy might be dominated by low-mass stars or half of galaxy mass comes from something dark, called dark matter.