Lifespans of Globular Clusters [20 marks]

a) Over long periods of time, open clusters gradually dissipate. Briefly explain why this is the case. **[2 marks]**

   Tidal forces gradually disrupt the cohesion of the cluster, especially members far from the center. This is worsened by the low mass of open clusters in general, resulting in weak gravitational forces holding the members together.

Due to the large number of stars in a typical cluster, the evaporation time of any cluster must be estimated through numerical simulations. In order to simplify the computations involved, these simulations are done assuming a spherical isolated cluster containing stars of equal mass [with aggregate mass equal to the mass of the actual cluster]. These "stars" are then distributed uniformly over the radius of the actual cluster. By doing multiple runs and observing how long these isolated model clusters take to evaporate, the cluster evaporation time can be estimated.

Using this methodology, a team of researchers have compiled the cluster evaporation time, t_{evap}, for the following globular clusters. [Data collected from http://gclusters.altervista.org/index.php]

Cluster Name	t_{evap} / 10^{11} years	lg[t_{evap}]	M_V	Distance from galactic center/kpc**	Distance to Sun/kpc
Omega Centauri	12.30	12.09	-10.26	6.4	5.2
M2	2.51	11.4	-9.03	10.4	11.5
M3	6.16	11.79	-8.88	12	10.2
M4	0.85	10.93	-7.19	5.9	2.2
M5	2.57	11.41	-8.81	6.2	7.5
M10	0.79	10.9	-7.48	4.6	4.4
M12	0.74	10.87	-7.31	4.5	4.8
M13	2.00	11.3	-8.55	8.4	7.1
M14	2.45	11.39	-9.1	4	9.3
M15	2.09	11.32	-9.19	10.4	10.4
M19	2.39	11.38	-9.13	1.7	8.8
M22	1.70	11.23	-8.5	4.9	3.2
M28	1.48	11.17	-8.16	2.7	5.5
M30	0.76	10.88	-7.45	7.1	8.1

*The stated absolute magnitudes have been corrected for interstellar extinction.
**Proxy for semi-major axes of these globular clusters.

It is hypothesized that the evaporation time for each i^{th} cluster is related to cluster mass by the following formula.

$$t_{evap,i} = k_1 M^{\gamma} \varepsilon_i$$

Where k_1 is an unknown constant, γ is a parameter of interest to be estimated, M is the total cluster mass, and ε_i is an error term specific to each cluster. This error term is essentially an adjustment for each individual cluster due to differences in physical characteristics [aka factors other than mass]. Since these error terms are assumed to be uncorrelated with cluster mass, you can treat them as constants for the rest of the question.

b) An astronomer suspects that γ is larger than 0. Using your knowledge of astrophysics, explain intuitively why this is likely to be the case **[3 marks]**

   As m increases, the gravitational force holding each member within the cluster increases, holding all else constant [1 mark]. Hence, members have greater difficulty escaping, increasing the evaporation time. [1 mark] Hence, t_{evap} must increase as M increases, which only occurs if γ is larger than 0. [1 mark]

However, estimates for a globular cluster's mass often contain significant measurement error. As an alternative, an astronomer suggests testing the following hypothesis instead.

$$t_{evap} = k_2 L^{\gamma} \varepsilon_i$$

With L being the globular cluster's luminosity.

c) What critical assumption is required for these two equations [and by extension hypotheses] to be equivalent? [2 marks]

$$L \propto M$$

   In words, the ratio between observed stellar mass to the total cluster mass is nearly the same for all clusters

d) Assume that your critical assumption holds. Through a suitable transformation, propose how you would test the hypothesis. Your answer should include an equation to be estimated by data analysis. **[3 marks]**

Log/ln the second equation [1 mark]

$$\log t_{evap} = \log k_2 L^{\gamma} \varepsilon_i \quad = \log k_2 \varepsilon_i + \gamma \log L$$

What is log L? Recall $\frac{L}{Lsun} = 10^{\frac{Msun - Mv}{2.5}}$. Log and simplify to get $\log L = \log Lsun + \frac{Msun}{2.5} + \frac{Mv}{2.5}$ [1 mark]

Substitute and simplify to get the final equation [.5]

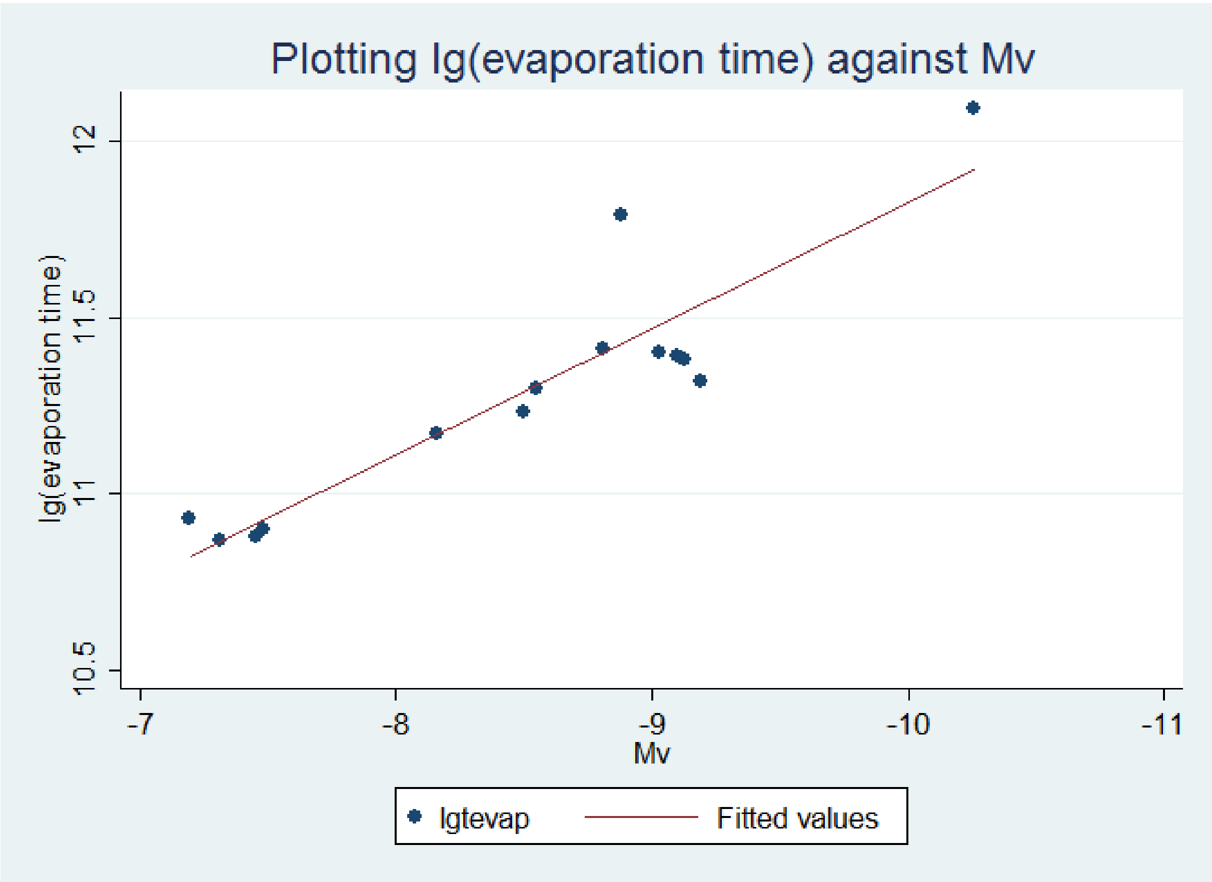
$$\log t_{evap} = \log k_2 \varepsilon_i + \gamma[\log Lsun + \frac{Msun}{2.5} + \frac{Mv}{2.5}]$$

$$\log t_{evap} = \log k_2 \varepsilon_i + \gamma \log Lsun + \gamma \frac{Msun}{2.5} + \frac{\gamma}{2.5} M_v$$

Hence, test hypothesis by plotting log [or ln] t_{evap} against M_v. The result should be a straight line. [0.5]

e) Using the data, plot your estimated equation on a suitable graph. Comment on the validity of the above hypothesis. **[6 marks]**

4 marks for accurate plots.



Good fit. The hypothesis seems to describe the data well. [2 marks]

Stata output for the dataset [not required]:

lgtevap	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf. Interval]	
Mv	-.358406	.0430717	-8.32	0.000	-.4522511	-.2645608
_cons	8.242525	.3414936	24.14	0.000	7.498474	8.986576

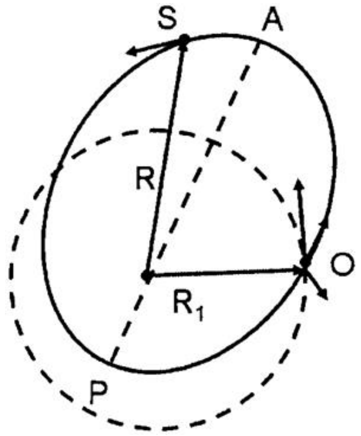
R-squared = 0.8330

f) Which type of clusters are most likely to evaporate before their predicted evaporation time? Give some examples from the sample above and briefly explain your reasoning. **[4 marks]**

GCs that **orbit close to the galactic center/have small semi-major axes** are most likely to evaporate before what this model predicts (e.g. **M19 and M28**). [2 marks]. The close proximity to the massive galactic center is likely to lead to strong tidal forces being exerted on the above globular clusters [1 mark].These tidal forces will gradually strip outlying stars away from these close globular clusters, accelerating their evaporation. [1 mark]

Orbital Mechanics (20 marks)

Satellite A moves in a circular orbit around Earth with radius R_1, period T and possessing a mass m. A moment later a small rocket on the satellite is turned on to change its direction so that it becomes an ellipse. These changes makes the satellite to loss half its angular momentum but its period and total energy remains constant. See the diagram below.



a) Show that the orbital velocity at any point in the new orbit is given by:

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$ **[4 marks]**

*r denotes distance from the center of Earth

b) What is the distance of the nearest point (perigee, P) and the farthest point (apogee, A) the satellite from the center of the Earth, expressed as a function of R_1? **[5 marks]**

c) Hence or otherwise, determine the eccentricity of the final orbit. **[2 marks]**

d) Now, suppose that the satellite remains in this specific orbit at its end of its useful life. Due to atmospheric drag, the magnitude of its total energy increases by 1% over the course of a year. Calculate the resultant change in the semi-major axis and its new perigee, supposing no change in eccentricity. **[3 marks]**

e) Instead of letting satellite A's orbit decay uncontrollably, the owner decides to fire the satellite's rocket at point P. This causes the satellite to lose all of its kinetic energy and engage in free-fall. How long does it take for the satellite to fall to Earth? Express your answer in terms of T. For simplicity, you may treat the Earth as a point. **[6 marks]**

Answers

a) The total energy of the satellite is conserved throughout the orbit. Furthermore, since the period is unchanged, the semi-major axis of the new orbit is also unchanged. (Indeed, participants should know that the total orbital energy is purely a function of the semi-major axis

-> same total energy implies same semi-major axis)

$$\text{TE} = -\frac{GMm}{2r_1} = -\frac{GMm}{2a}$$

Now we consider an eccentric orbit. When $r \neq a$, GPE is (still) given by :

$$\text{U} = -\frac{GMm}{r}$$

The kinetic energy of the satellite is then:

$$\text{KE} = \text{TE} - \text{U} = -\frac{GMm}{2a} - \left(-\frac{GMm}{r}\right) = GMm\left(\frac{1}{r} - \frac{1}{2a}\right)$$

Substitute the formula for KE and solving for v:

$$0.5mv^2 = GMm\left(\frac{1}{r} - \frac{1}{2a}\right)$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

b) Index "1" indicates the initial orbit condition and index "2" indicates final orbit condition

Angular momentum:

L2 = 0.5 L1

m*V2*r2 = 0.5*m*V1*r1

V2 = 0.5*V1*r1/r2 …………………………………………………. (Eq. 1)

Circular orbit :

Fg = m*a_s

GMm/(r1)^2 = m(V1)^2/r1

V1 = sqrt(GM/r1) …………………………………………………. (Eq. 2)

Energy equation :

K1 + U1 = TE1

0.5*m*(V1)^2 – GMm/r1 = TE1 ➔ (Substitute eq. 2)

E1 = - GMm/r1 …………………………………………………. (Eq. 3)

K2 + U2 = TE2

K2 + U2 = TE1 ➔ Total energy remains constant

0.5*m*(V2)^2 – GMm/r2 = TE1 ➔ (Substitute eq. 1,2,3)

4(r2)^2 – 8r1r2 + 4(r1)^2 = 0 ➔ using square formula

r2 = (1 ± sqrt(3)/2)r1

r apogee = (1 + sqrt(3)/2)r1

r perigee = (1 - sqrt(3)/2)r1

c) e = (r apogee – r perigee) / (r apogee + r perigee) = sqrt(3)/2

d)

$$\text{Initial TE} = -\frac{GMm}{2a_1}$$

$$\text{Final TE} = -\frac{GMm}{2a_2} = -1.01 \times \text{Initial TE}$$

Dividing one by the other:

$$\frac{\text{Initial TE}}{\text{Final TE}} = \frac{1}{1.01} = \frac{a_2}{a_1}$$

$$a_2 = \frac{a_1}{1.01}$$

The change is then simply $a_2 - a_1 \approx 9.91 \times 10^{-3} a_1$

The final perigee is then $a_2(1-\varepsilon) \approx 0.1326\, a_1$

e) We can describe an object in free fall as simply following an “orbit” with perigee = 0 and apogee = P = $(1 - \frac{\sqrt{3}}{2})a_1$. This is the most important insight.

The new semi-major axis is then half of P = $(0.5 - \frac{\sqrt{3}}{4})a_1$

Using Kepler’s third law, the period of the new orbit is:

$$T_{new} = \left[\frac{\left(0.5 - \frac{\sqrt{3}}{4}\right)a_1}{a_1}\right]^{1.5} T = \left(0.5 - \frac{\sqrt{3}}{4}\right)^{1.5} T$$

The time it takes to fall to Earth is then the time it takes to travel from apogee to perigee (since we can treat the Earth as a point). This is just half the period, so:

$$T_{free\ fall} = 0.5\, T_{new} = \frac{\left(0.5 - \frac{\sqrt{3}}{4}\right)^{1.5}}{2} T$$

A clash of two stars (20 marks)

The Death Star has been chasing several capital ships from the retreating Rebel Alliance. All that travel has depleted its fuel cells however, and the Death Star has had to call off the chase and replenish its energy supplies near Antares.

As an astronomer specialising in the field of astroseismology, obviously you take the opportunity to scan Antares with your instruments while you approach it. Astroseismology involves the study of a star's oscillation frequencies in order to probe its internal structure. As different waves penetrate into different layers of the star, looking for the presence of certain waves and their timing can shed some light about the density profile of the star and its internal structure.

After asking your research assistant to load all the data you have gathered into your personal supercomputer, you take some time to prepare a cup of tea. When you return a few minutes later, the supercomputer has completed its work and is printing out its results.

You scan through the printouts. You chance upon this line:

"Silicon fusion shell: PRESENT"

You spit out your tea, most of it landing on your hapless assistant.

a) Your displeased assistant takes a look at the printout and rages: "What is so special about this 'Silicon fusion shell' that requires you to spit tea all over ME?!"

   Explain to your nonplussed assistant the significance of your results. In particular, what's coming up in the very near future? Given the mutinous look on his face, you are advised to limit yourself to a few sentences. **[1 mark]**

   As the final stage of fusion, Silicon burning signifies the **imminent end of Antares**.
   At a certain point, the **core collapses,** triggering a Type II **supernova**

"There isn't too much time. Let Emperor Palpatine know. We must change course immediately"

As your assistant scurries away, you recall some details about silicon fusion.
Silicon fusion commences at a temperature of 2.7 billion Kelvins in a tiny shell of approximately 0.003 solar radii. While the whole silicon fusion shell is a frothing mess that undergoes complex reactions, for simplicity, we may approximate these fusion processes by the overall reaction:

$$^{28}_{14}Si \ + \ ^{28}_{14}Si \ \rightarrow \ ^{56}_{28}Ni + \gamma \ (10.914 \ Mev)$$

b) After further simulations with your personal supercomputer, you find that a net total of 10^{36} J of EM radiation plus heat is transferred to the shell's surroundings every second. Hence, estimate the time left before core collapse. **[4 marks]**

Assume the following:

- All of this energy is supplied through the previous overall reaction with 100% efficiency.
- Heat and EM radiation are the only significant means of energy transfer: there are no other means in which thermal energy can be lost from the shell.
- $^{56}_{28}Ni$ has a mass of 56 amu.
- Initially, Antares does not possess any iron/nickel core.
- All nickel produced in the silicon fusion shell immediately sinks to form an iron/nickel core.

Core collapse occurs when the core of the star reaches 1.5 solar masses

$$\text{Ni nuclei produced every second:} \frac{10^{36}}{10.914 \times 1.6 \times 10^{-13}} = 5.72 \times 10^{47}/s$$

$$\text{Mass accumulating every second: } 5.72 \times 10^{47} \times 56 \times 1.6605 \times 10^{-27}$$

$$= 5.32 \times 10^{22}\ kg/s$$

$$\text{Time taken: } \left(1.5 \times 1.9891 \times 10^{30}\right) \div 5.32 \times 10^{22} = 5.6 \times 10^{7} s \approx 649\ days$$

c) When it is finally formed, simulations suggest that the radius of the 1.5 solar mass iron-nickel core reaches 6000 km. Shortly after reaching this critical point, the entire core collapses into a neutron star of radius 30 km within seconds. Calculate the energy released by core collapse. **[1 mark]**

$$Energy\ released\ by\ core\ collapse = -\Delta U = -\frac{3GM^2}{5}\left(\frac{1}{6000km} - \frac{1}{30km}\right) = 1.182 \times 10^{46} J$$

It occurs to you that the power of the onboard Death Star laser exceeds $10^{33}\ W$, enough to pulverise Alderaan into flying dust in less than a second. Yet the energy released in Antares' death throes makes the Death Star looks positively weak in comparison.

Recalling your work, you know that most of this energy is converted into neutrinos, particles that rarely interact with matter. Hence, the greatest threat from Antares lies from its EM blast.

d) Only 1/10 000 of this stupendous amount of energy is converted into EM radiation, of which 1/50 000 is isotropically emitted in the initial blast wave. While the Death Star is implausibly well-prepared with an army of engineers, shields and heat dissipating radiators, the Death Star will suffer catastrophic hull failure when the power of incident radiation exceeds **5 MJ/m²**.

What is the minimum radius of the safe zone? In other words, how far must this near invulnerable Death Star be from Antares for it to survive? Express your answer in terms of light hours. **[3 marks]**

Energy of peak blast wave = $\frac{1.182\times10^{46} J}{10\ 000 \times 50\ 000} = 2.364 \times 10^{37} J$
To get an intensity of 5 MJ /m^2, this energy must be spread out over a surface area of 4.728×10^{30} square meters.
The radius of a sphere with this surface area is then $\sqrt{\frac{4.728\times10^{30}}{4\pi}} = 6.134 \times 10^{14}\ m = 567.95\ light\ hours$

NB: The radius of the Death Star is negligible compared to the distance of the safe zone, so we can approximate the Death Star as an essentially flat sheet. Naturally, this wouldn't hold if the results were *much much much much much* smaller.

Glancing over at the monitor, you note that the Death Star is already 500 light hours away from Antares. For perspective, that's more than 3600 AU!

"Whew! Soon, we'll be safely be out of reach."
Or so you thought.
The silence of your office is suddenly interrupted by a cacophony of pings, each one signifying a neutrino detection. Amidst the chaos, you immediately alert the flight deck to accelerate ASAP.

Emperor Palpatine is highly displeased and seems ready to kill you with force lightning at any moment. However, he decides to save his own skin first.

e) Your estimate of the time of core collapse has proven to be way too optimistic. You really should have checked if your answer felt right!

Name and briefly explain one process that we have neglected in our calculations, leading to this overestimation. **[2 marks]**

-Photodisintegration of nuclei as gamma rays are absorbed by nuclei, triggering decay
-Pair production as gamma rays are absorbed by nuclei to create particle-antiparticle pairs
-Neutrinos produced during nuclear reactions carry away significant amounts of energy.
-Neutronization as protons and electrons are forced together under these extreme conditions.
(any one of the above)

NB: Participants were expected to name reactions that removed energy from the shell. Answers that invoked processes that produced energy (like the CNO cycle) were not accepted.

The engine shudders with an almighty noise and you are tossed backward by a tremendous force as the ship accelerates. You almost blank out under the pain. Could this be a terrible dream?

When you recover, you see on your monitor that the Death Star is now moving at its top speed of 0.9c. Thankfully, your simulations suggest that the EM blast wave emerges 3 hours after the initial neutrino pulse. You frantically hurry to determine your fate…

f) To an observer at rest in the safe zone, how much time does it take for the Death Star to reach the safe zone? **[1 mark]** How about for the blast wave? **[1 mark]** Express your answers in hours.

$$\text{Time taken for Death Star} = \frac{6.134\times10^{14}\ m - 500\ light\ hours}{0.9c} = \frac{7.38\times10^{13}\ m}{0.9c} = 75.9\ \text{hours.}$$

$$\text{Time taken for blast wave} = \frac{7.38\times10^{13}\ m}{c} + 3\ hours = 71.3\ \text{hours.}$$

g) Your assistant points out that you forgot to consider the effects of special relativity, which are significant at these speeds. Determine the amount of time (in hours) it takes for the ship to travel to the safe zone in the ship's point of view. **[3 marks]**

$$\text{Length contraction} = \frac{7.38\times10^{13}\ m}{\gamma} = \sqrt{1-0.9^2}(7.38\times10^{13}\ m) = 3.217\times10^{13}\ m$$

$$\text{Apparent time taken} = \frac{3.217\times10^{13}\ m}{0.9c} = 33.09\ hours$$

h) Your assistant comments: "It seems like if you apply classical physics, we are doomed. But if we apply Special Relativity, we seem to end up outpacing the blast wave (as calculated in f). So which should I believe? Or have we done something wrong?"

Why do parts f and g suggest such wildly different outcomes? In other words, should we reject the answers of one, both or neither?
Hence, determine whether the Death Star lives to die another day. Your answer determines the lives of many! **[4 marks]**

The discrepancy arises because **we haven't considered how much time it would take for light to travel to the safe zone** in the Death Star's POV. **According to special relativity, the speed of light is the same in all inertial reference frames**. Hence, we must **also account for the fact that light will travel along the length contracted distance in g**. This means the implied conclusion **in part f is correct**, and you should start panicking!

780 days of work (20 marks)

An astrophotographer has been taking astronomical images of several objects over the past 780 days with each observation session separated by 20 days.

a) A fellow astronomer suggests the use of a Dobsonian. Explain why this is bad advice in astrophotography. **[2 marks]**

During these sessions, he has meticulously kept track of the positions of two planets. Unfortunately, he has lost the accompanying notes and now cannot determine which planet corresponds to the planet in his observation log. The data is reproduced below:

Observation Period (days)	Planet 1	Planet 2
	$RA-RA_{sun}$/hr	$RA-RA_{sun}$/hr
0	-1.35389	1.52806
20	-0.95917	1.15556
40	-0.57028	0.82472
60	-0.23056	0.53278
80	0.00753	0.25861
100	0.39139	-0.02417
120	0.73750	-0.32167
140	1.15194	-0.67444
160	1.55694	-1.05694
180	1.87583	-1.43944
200	2.09750	-1.79556
220	2.27639	-2.12194
240	2.47750	-2.44222
260	2.74194	-2.79361
280	3.05333	-3.22000
300	3.29028	-3.76000
320	3.21556	-4.43694
340	2.50194	-5.24556
360	0.76944	-6.14472
380	-1.43000	-7.09917
400	-2.65472	-8.11694
420	-3.00139	-9.29250
440	-2.94778	-10.7317
460	-2.78056	-12.40722
480	-2.62861	-14.06644
500	-2.50944	-15.49467
520	-2.36944	-16.67611
540	-2.13028	-17.64889
560	-1.75861	-18.42644

580	-1.31000	-19.01389
600	-0.87722	-19.42944
620	-0.50750	-19.70922
640	-0.18917	-19.90856
660	0.11917	-20.09754
680	0.45667	-20.34467
700	0.82500	-20.68967
720	1.16889	-21.11689
740	1.43722	-21.56500
760	1.64500	-21.97611
780	1.85194	-22.32811

b) A student claims that $RA - RA_{sun}$ measures how many hours the planet can be seen after sunset. Is this true? **[3 marks]**

c) What does the term synodic period mean? Hence or otherwise, derive a formula for the synodic period of a planet relative to Earth **[5 marks]**

d) For each planet, plot $RA - RA_{sun}$ over time. Hence or otherwise, estimate the synodic period for each planet. Also, find the observation period(s) when each planet reaches greatest elongation(s) **[8 marks]**

e) Which planets in our solar system are likely to correspond to Planet 1 and Planet 2? Justify your answer. **[2 marks]**

Answer :

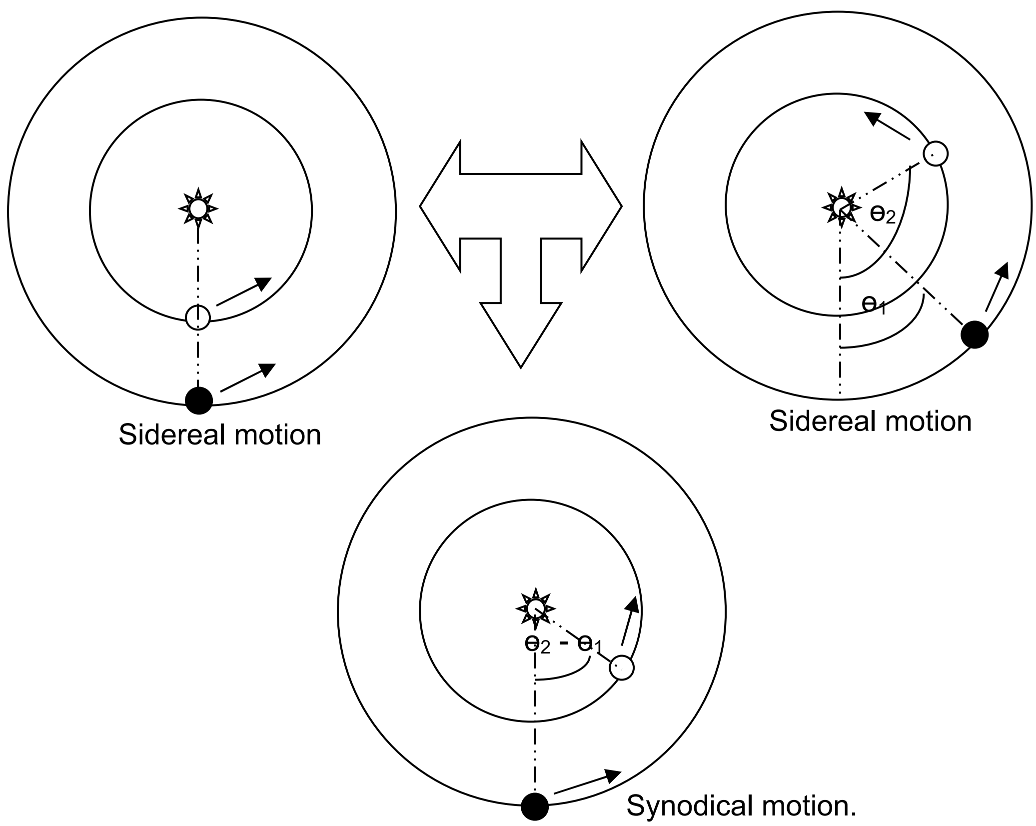
a. Dobsonian telescope is an telescope which uses alt-azimuthal mounting type and Newtonian Optical System. Since it is an alt-azimuthal telescope, it has to move in two axes (Azimuth and Altitude) in order to track a sky object and take a photo of it because the reference frame is observer's horizon. Those are the drawbacks in using alt-azimuthal telescope for astrophotography purposes since sky objects are moving with respect to Earth's Equator. Equatorial type telescopes are highly recommended for astrophotography imaging because those telescopes will only need to move in 1 axis which is Hour Angle axis or Right Ascension axis.

b. It is important to note that RA gets larger in magnitude towards east direction. Now, let's take a look for each case :

- When $0h \leq RA_{object} - RA_{sun} < 12h$, it is true that $RA_{object} - RA_{sun}$ measures how many hours the planet can be seen after sunset.
- When $12h \leq RA_{object} - RA_{sun} < 24h$, $24h - (RA_{object} - RA_{sun})$ measures how many hours the planet can be seen after sunset.

Conclusion : It is not necessarily true that $RA - RA_{sun}$ measures the observable period of that object. (Note that observable period is measured from the time the object rises until it sets and also after sunset and before sunrise)
Bonus: The above considerations only apply at the equator. To see why this isn't true generally, consider an observer at the North Pole. If the planet has a slight northerly declination, it will be visible all night if the sun is slightly below the horizon.

c. Synodic period is defined as the period needed for the planets to return to the same or proximately the same position relative to the sun as seen by an observer on the Earth.

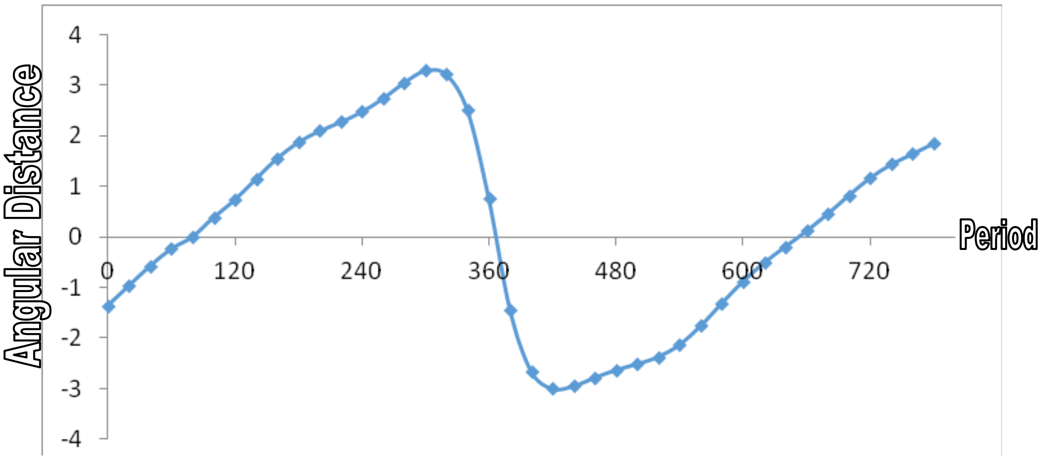


Let’s define :
w = angular velocity
P1 = Sidereal period of “Black” Planet
P2 = Sidereal period of “White” Planet
P   = Synodic period

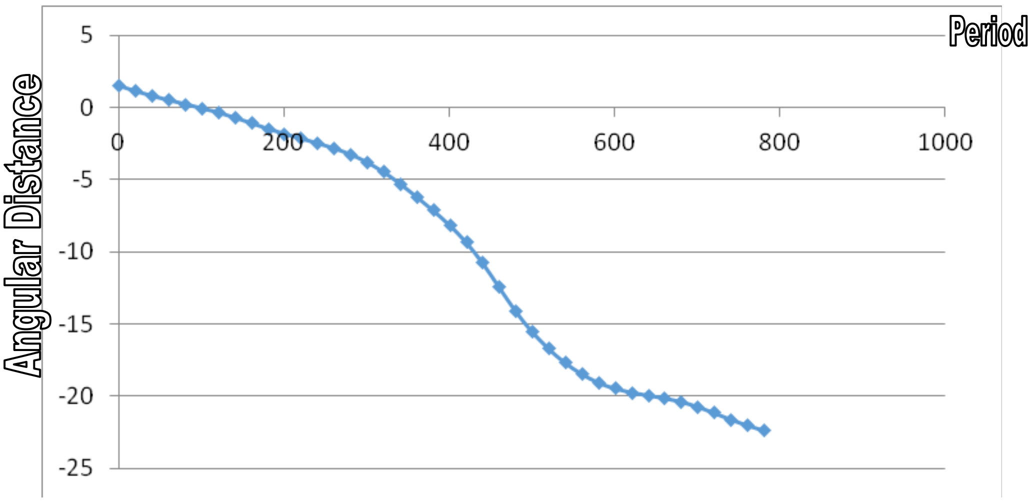
(w2-w1)*P = 360o
360o/P2 – 360o/P1 = 360o/P
P = [1/P2 – 1/P1]^(-1)   →   For inferior Planet

P = [1/P1 – 1/P2]^(-1)   →   For Superior Planet

d. Planet 1 :



Planet 3 :



For planet 1, the synodic period is the period between two consecutive peaks or two consecutive troughs.
Planet 1 : Synodic period = 29 consecutive 20-days observation interval = 580 days

For planet 2, since it does not form sinusoidal wave, then the synodic period is the period to get back to the same angular distance. In this case, from the graph, the length of period from angular distance equals to 0h to angular distance equals to -24h (0h).

Planet 2 : Synodic period ~ 39 consecutive 20-days observation interval = 780 days.

Planet 1 reached its greatest elongation as seen from the Earth on the 320th day of observation with angle equals to 48.23340o (3.21556h) and on the 420th day of observation with angle equals to -45.02085o (-3.00139h)

Planet 2 reached its approximate greatest elongation as seen from the Earth on the 460th day of observation with angle equals to 186.11558o (-12.40772h)

e. As we analyze the graph for planet 1, the graph has sinusoidal shape. This figure can only occur for inferior planets which are Mercury and Venus. We can calculate directly the synodic period of Mercury and Venus using this following formula :

$P_{syn} = [1/P_{revolution.Planet} - 1/P_{revolution.Earth}]^{(-1)}$

$P_{syn,Mer} = [1/88 – 1/365.25]^{(-1)} = 115.93$ days

$P_{syn.Ven} = [1/225 – 1/365.25]^{(-1)} = 585.96$ days

Hence, Planet 1 is Venus.

For planet 2, it must be superior planet since only superior planet can have oppositions (as seen by $RA-RA_{sun}$ going to 12 hours)

For superior planet, we can calculate the synodic period using this following formula :

$P_{syn} = [1/P_{revolution.Earth} - 1/P_{revolution.Planet}]^{(-1)}$

$P_{syn,Mars} = [1/365.25 – 1/687]^{(-1)} = 779.88$ days

$P_{syn,Jupiter} = [1/365.25 – 1/(12*365.35)]^{(-1)} = 398.46$ days

$P_{syn,Saturnus} = [1/365.25 – 1/(29.5*365.35)]^{(-1)} = 378.07$ days

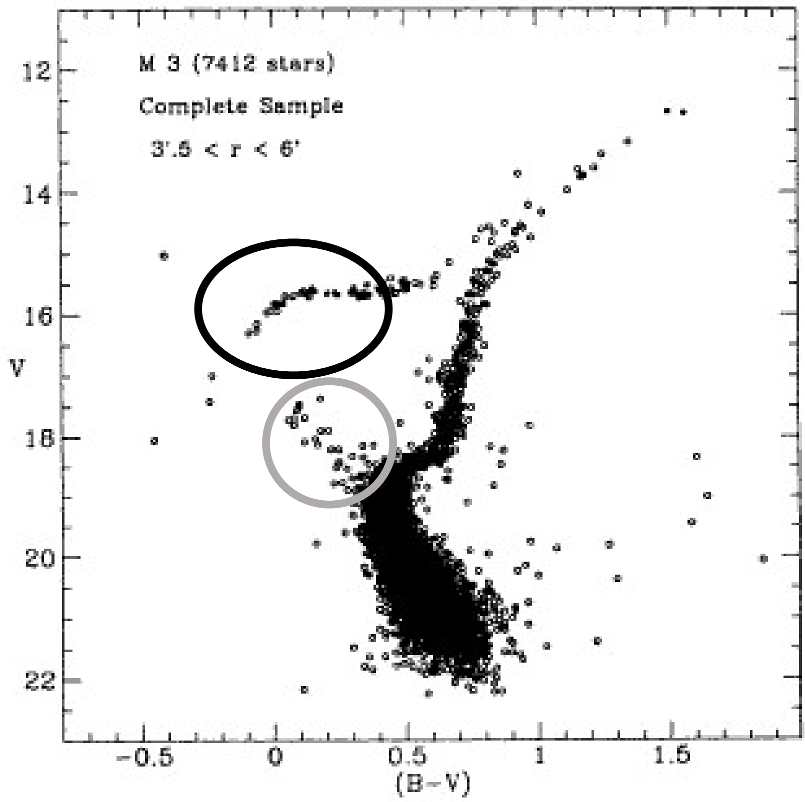
$P_{syn,Uranus} = [1/365.25 – 1/(84*365.35)]^{(-1)} = 369.65$ days

$P_{syn,Neptunus} = [1/365.25 – 1/(165*365.35)]^{(-1)} = 367.48$ days

Hence, Planet 2 is Mars.

M3 (20 marks)

The diagram below shows the color-magnitude diagram for M3. Located 22,500 light years away in the constellation Canes Venatici, it is one of the brightest globulars in the night sky.



a) What are the stars in the bottom circle categorised as? **[1 mark]** Explain how they are believed to have formed and how it leads to their unusual position on this diagram. **[2 marks]**

Blue stragglers. Mass transfer/mergers increases the luminosity of these stars by adding fuel, effectively turning back the clock.

b) What are the stars in the top oval categorised as? **[1 mark]** Explain how they are believed to have formed and how it leads to their unusual position on this diagram. **[2 marks]**

Horizontal Branch stars. They are formed from red giants that have just undergone the helium flash/helium burning. During this process, their surfaces contract and increase in temperature, keeping their luminosity nearly constant while becoming bluer.

c) What is the apparent magnitude of stars at the main sequence turn-off point? **[1 mark]** How about their absolute magnitude and luminosity? **[3 marks]**

Accept 18.5 – 19
We'll take 19 for the rest of this answer. (1 mark)

22500 ly = 6900 pc
The distance modulus can then be shown to be 14.19 (1 mark)
Absolute magnitude is then simply the difference, aka 19-14.19 = 4.805 (0.5 mark)

$$\frac{L}{L_{sun}} = 10^{\frac{4.7554-4.805}{2.5}}$$
$$L = 0.9547 \text{ solar luminosities}$$

(Accept the use of the bolometric formula as given in the Formula Book)

d) If the mass-luminosity relation holds true, what is the approximate relationship between a star's mass and its lifetime on the main sequence? Briefly explain your reasoning and express your answer in the form of $T \propto M^x$, where x is a value to be derived with suitable mathematical workings. **[4 marks]**

Lifetime of a star is proportional to the amount of fuel it has, divided by the rate of fuel consumption (2 marks). Hence T proportional to M/L (1 mark)

Sub in the mass-luminosity relation to get T proportional to $M^{-2.5}$ (1 mark)

e) Hence, estimate the age of M3, assuming that all the stars in M3 formed at the same time. For reference, the main sequence lifetime of our sun is 10 billion years. **[2 marks]**

By the mass-luminosity relation, the mass of the stars at the main sequence turn-off is:

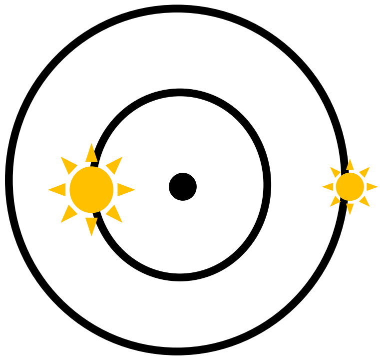
$$M = 0.9547^{\frac{1}{3.5}} \text{ solar masses} = 0.9868 \text{ solar masses}$$
$$T = 10^{10} \times 0.9868^{-2.5} = 1.03 \times 10^{10} \text{ years}$$

Due to the density of globular clusters, many stars in M3 are actually binary star systems.

f) Many of these binary star systems consist of a main sequence star and a white dwarf/neutron star. Presumably, these binary star systems were initially both main sequence stars (with masses m and M). The heavier star (with mass M) subsequently evolved faster, losing significant amounts of mass and leaving behind a stellar remnant.

How much mass can the heavier star lose before the system becomes unbound? Express your answer in terms of the total initial mass of the system. To simplify your calculations, you may assume a circular orbit. **[4 marks]**

Let the initial mass of the massive star be M1. Now consider the initial setup: two stars (of different masses) orbiting a common centre of mass.



Now, let D represent the distance between the two stars and r represent the distance of the less massive star from the centre of mass

We can then find a relationship between r and D by utilising the formula for the centre of mass (1 mark)

$$M_1(D - r) = mr$$
$$r = \frac{M_1}{M_1 + m} D$$

Now, the initial orbital velocity of the outer (and less massive) star can be found by equating the gravitational force between the two stars and centripetal force. (1 mark)

$$\frac{GM_1 m}{D^2} = \frac{mv^2}{r}$$
$$\frac{GM_1 m}{D^2} = \frac{M_1 + m}{M_1 D} mv^2$$
$$\frac{G(M_1 + m)}{D} = v^2$$

$$v = \sqrt{\frac{G(M_1 + m)}{D}}$$

Now, when the massive star dies, it sheds mass, leaving behind a remnant with mass M2. The escape velocity at distance D then can be found (1 mark) when the total energy exceeds 0 (aka, KE + GPE =0)

$$0.5mv_{escape}{}^2 - \frac{GM_2 m}{D} \geq 0$$
$$v_{escape} \geq \sqrt{\frac{2GM_2}{D}}$$

Thus, for the system to remain bound (1 mark):

$$v < v_{escape}$$
$$\sqrt{\frac{G(M_1 + m)}{D}} < \sqrt{\frac{2GM_2}{D}}$$
$$(M_1 + m) < 2M_2$$
$$M_2 > 0.5(M_1 + m)$$

You'll find that the star cannot lose more than half the total initial mass of the system.