AstroChallenge 2022
Senior Team Round

SOLUTIONS

Saturday 4th June 2022

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper consists of **33** printed pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your name, school, and team.
6. It is your responsibility to ensure that your answer script has been submitted.
7. The marks for each question are given in brackets in the right margin, like such: [**2**].
8. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).

Question 1 Short Answer Questions

An Evening Date

Analyse the story below and answer the following questions.

One clear December evening, Jack brought Jill to a rooftop of a high-rise building to stargaze. Once settled down, Jack started setting up his telescope.

"Help me hold this," Jack said as she passed Jill the OTA.
"What is this?" Jill asked as she reached out to take the OTA.
"It's the optical tube assembly of a refractor telescope, careful not to drop it as the glass inside might crack," Jack explained.

Thereafter, Jack tightened the OTA onto the mount and inserted the eyepiece. He carefully balanced the setup before mounting and aligning the finder scope.

Looking at his watch, Jack realised that it was only 8pm and Comet Leonard was still above the horizon. He quickly pointed his telescope at the comet hoping to catch a glimpse of it before it sets. After 3 minutes of intense searching, Jack exclaimed: "I found it! Come and take a look through the eyepiece!"
"What am I looking at?" Jill questioned with a puzzled look on her face. "Is it that super bright thing near the horizon there?"
"Nope, that bright thing is a planet, it's the brightest planet in the night sky, with a magnitude of 5. But what you are seeing right now is a comet, it has a magnitude of -4." Jack explained patiently.
"Wow! I can't even see the comet in the sky, how were you able to find it and even tell that it is a comet?" Jill asked.

"Well it's quite simple and I shall leave this question as an exercise for the readers." Jack muffled to himself, not wanting Jill to find out that they were just characters in a story.

"What did you say?" "Oh, I was saying I just moved the two axes of the mount up and down, left and right and just happen to point my telescope at it. It's quite simple to use and has no counterweights. Hahahaha." He explained as he tried to hide the awkwardness with laughter.
"I see." Jill mumbled as she stares at the comet as it slowly drifted out of the view of the eyepiece. "Why is the comet moving out? Is it because it is very close to us?"
"It's because the Earth is rotating. The comet itself is quite far, although not as far as the background stars, and the movement is minute," Jack said.

"Let me change to an eyepiece of a shorter focal length so that the comet stays longer in the field of view."
"It's alright, I've seen enough of it." Said Jill as she seemed unimpressed by the comet.

"Alright, let's take a look at the winter night sky instead!" Jack said as he tried to reignite her interest.

(a) **Name and explain three factural errors/mistakes in the above paragraphs.** **[3]**

Solution:

1. Jack balanced the setup before mounting the finder scope. This will cause the setup to be unbalanced again.
2. The magnitude of the comet and the brightest planet is flipped. You won't be able to see a magnitude 5 object easily with the naked eyes.
3. Jack said that he used an eyepiece of a shorter focal length to make the comet stay longer in the Field of View (FOV). However, a shorter focal length means larger magnification and the comet will stay shorter in the FOV.

Note: The comet moving out of the FOV due to its own movement is not a valid explanation. This is because the apparent movement of the comet due to its motion through space is negligible when compared to the comet's apparent movement due to Earth's rotation.

(b) **What was the planet near the horizon?** **[1]**

Solution:

Venus. It is the brightest planet in the night sky.

(c) **What type of mount was used? Explain your answer.** **[2]**

Solution:

The mount used is an altitude-azimuth mount.

This is due to the lack of a counterweight and the up-down-left-right movement of the axes.

(d) **Explain how Jack was able to find and differentiate the Comet from other stars.** **[2]**

Solution:

Jack was able to find the Comet by checking Stellarium/memorising the location of the comet/star hop from nearby stars/any reasonable answers.
Jack was able to differentiate the Comet as the Comet has a tail/Comet moves with respect to background stars.

Note: Jack would not have been able to use the Comet's brightness as a differentiating point since it is too dim (mag 5).

(e) **Without using radar ranging, suggest a way that the distance of the comet can be calculated or measured.** **[2]**

Solution:

We can measure the perihelion and aphelion of the Comet. Calculate the period using Kepler's Third Law. Then calculate the location of its orbit using Kepler's Second Law. Then we will get the distance by taking the angle between the Sun and Comet, combining with the location of the orbits.

We can use parallax by taking measurement of the parallax angle from two different points on Earth, thus getting the distance.

Note: Measuring and comparing the comet's parallax over a few will not yield a unique result. Since we do not know its velocity, it could be very far but fast-moving or very near but slow-moving.

(Story Cont.) After that, the two of them sat down and started to look up into the night sky. Jack explained to Jill how her zodiac constellation looked like and how to recognise the alpha star of that constellation, which is also a red giant. Jack also showed Jill some other deep sky objects (DSOs) through his telescope.

(f) **Name 4 possible DSOs that were seen through the telescope. They must be of different types. Put the type of DSO in bracket beside the name.** **[2]**

Solution:

Any plausible non-summer sky DSO.

Examples from the Messier Catalogue: Great Orion Nebula (M42) [HII Emission Nebula], Pleiades (M45) [Open Cluster], Andromeda Galaxy (M31) [Galaxy], Little Dumbbell Nebula (M76) [Planetary Nebula]

(g) **Jill wondered how the red giant in her zodiac constellation was formed. Explain how the red giant was formed.** **[2]**

Solution:

When stars of similar mass to our Sun runs out of hydrogen in the core for fusion, it begins to fuse helium in the core and hydrogen in the shell around the core. This caused the star to expand into a red giant.

(Story Cont.) As the night continued, their necks got tired they lied down beside each other and continued their conversation.
"Hey look, it's the moon!" Jack pointed near the horizon at around midnight.
Jill got up and stared at the moon. "Wow it's so beautiful!"

(h) **What was the phase of the moon on that day? Explain your answer.** **[2]**

Solution:

The Moon phase was last quarter/third quarter. Moon moves eastwards in the night sky. Since the Moon rises at midnight, it is already three quarters through the lunar cycle.

You can also mention that the Moon was in its Waning Crescent/Gibbous which means the same thing as last/third quarter respectively.

(j) **Explain how the orbit of the Moon causes the phases** **[1]**

Solution:

The angle that the Moon reflects the Sun's light causes the phases. At any time the hemisphere of the Moon that faces the Sun is lit, but due to the Moon's relative position to the Earth, not the entire lit side of the Moon is visible.

(Story Cont.) At around 3am, they both saw a flash in the far North.
"Was that a shooting star?" She asked.
"Yes, it is the Ursid meteor shower, we are lucky to see one with the moon up being so bright above us." Jack said.
"Make sure to make a wish when you see one again!"

(k) **Around what constellation can the origin of the meteor shower be found?** [1]

Solution:

Ursa Minor, Ursa Major, Draco. Since the radiant point is somewhere near these three constellations.

Note: For completeness, the Ursids' radiant point is located in Ursa Minor, near β UMi (Kochab).

(m) **Explain how the meteor shower was formed.** [2]

Solution:

When a Comet/Asteroid travels across the Sun, it leaves behind a trail of dust, rock and ice. When Earth's orbit crosses into the trail left behind, the small dust, rock, and ice enter Earth's atmosphere. They burned up in the atmosphere to form a meteor shower.

(Story Cont.) They stayed up till sunrise before making their way back home. **(Story End.)**

(n) **Bonus: Do you think Jill will go out with Jack again for another stargazing session? Explain your answer.** [1]

Solution:

*There is **no** singular "solution" to having a successful relationship. This question was left to the interpretation of the reader.*
We here at Astrochallenge advocate for responsible social interaction.

Question 2 A Celestial Dance

How to Physics?

In Physics, there are generally two ways of solving any question. One is through the analysis of the forces acting on any object and applying Newton's Second Law on the said object. However, this approach is often problematic for complex problem involving multiple forces as it requires dealing with the addition of vectors (which is rather difficult).
The alternative is to analyse the problem by considering the total energy on a system. This approach avoids the complex vector addition that is required when considering forces. Thus, it is often easier to generalise for more complex problems by simply including the energy due to additional objects.

For this question, we will start by looking at the Sun-Earth System and solving it using the Energy approach before generalising it to any arbitrary two-star system. This is often done in analyses that involves binary stars, multiple star systems (such as star clusters) and accurate exoplanet/solar system planet analysis.

Part I Earth-Sun System

We can start by looking at the total energy of the Sun-Earth-System. Let M be the mass of the Sun and m be the mass of Earth. The total energy for such a system is often written as:

$$E_{total} = \frac{1}{2}m\vec{v}^2 - \frac{GMm}{|\vec{r}_{12}|}$$

where $r_{12} = \vec{r}_S - \vec{r}_E$ and $\vec{r}_S$ and $\vec{r}_E$ are the position vectors of the Sun and Earth respectively with respect to an origin point O. $|\vec{r}_{12}|$ represents the magnitude of the distance between the Sun and Earth whilst $\vec{v}$ is the velocity of the Earth relative to the Sun. **Figure 1** below illustrates the system that has just been described.

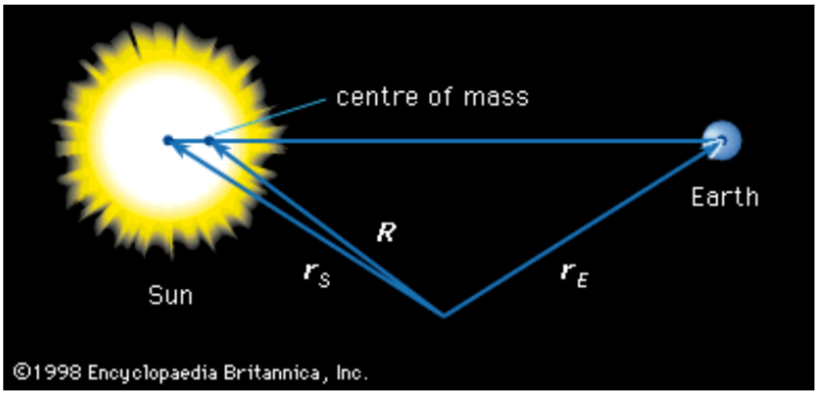

Figure 1: The Sun-Earth System

This total energy expression is written with the assumption that the sun is at the centre of the coordinate system and thus is not moving relative to the coordinate system. However, if we were to generalise this to an arbitrary two-star system, such an assumption **is no longer valid**.

The first step in generalising the approach for general two-star system is by taking an arbitrary co-ordinate system centre at some origin O as shown in **Figure 1**. In this frame of reference, both the Sun and Earth will have a velocity with respect to the coordinate system and thus an kinetic energy.

(a) **Rewrite the total energy of the Sun-Earth system in this new coordinate system, given the velocity of the Sun with respect to this coordinate frame is $\overrightarrow{V}$.** [1]

Solution:

The total (mechanical) energy of a system is the sum of the Kinetic Energies of each object in the system and the sum of the Gravitational Potential Energies due to the interactions between each pair of objects in the system.
For the Earth-Sun system, it is just:

$$\begin{aligned} E_{total} &= K_{\oplus} + K_{\odot} + U_{\odot\oplus} \\ &= \frac{1}{2}m\vec{v}^2 + \frac{1}{2}M\vec{V}^2 - \frac{GMm}{|\overrightarrow{r_{12}}|} \end{aligned}$$

Part II COM Frame

In actual orbital calculations, it is useful to use a particular coordinate system called the Centre of Mass (COM) frame. This is a special frame of reference for multiple-body systems. In this special frame, we are able to separate the often complex forces acting on any object within the system, such as the Earth, into two categories: Internal and External forces. Internal forces due to objects within the system are forces like the gravitational attraction between Sun and Earth, whilst external forces are forces due to objects outside the system of interest (like gravitational force due to other solar system object and due to the Milky Way).

This frame also allow us to analyse the complex motions of objects in this frame into two components: one component is the object's motion around the COM, which is purely due to internal forces and the second component is the motion of the COM (the system itself) due to external forces[1].

With this COM frame, we can now proceed to simplify our expression for the total energy obtained in **(a)**. To do this, we need to first determine the position of the centre of mass. This is given by the relation:

$$m\vec{r} + M\vec{R} = 0$$

$|\vec{r}|$ and $|\vec{R}|$ are the distances from the centre of mass to masses m and M respectively.

(b) **Find an expression for $\vec{r}$ and $\vec{R}$ in terms of $\overrightarrow{r_{12}}$, m and M. You are given the expression of the reduced mass is:**

$$\mu = \frac{Mm}{M+m}$$

[2]

Solution:

Note: Please note that there was a typo error in the original question paper sent out during AC Day 1 where the vector signs in the COM relation went missing. The following solution will show the answer for the corrected version as reflected above.
We aim to eliminate $\vec{R}$ from the expressions by introducing the object separation distance $\overrightarrow{r_{12}}$.

$$m\vec{r} + M(\vec{r} - \overrightarrow{r_{12}}) = 0$$

$$(m+M)\vec{r} = M\overrightarrow{r_{12}}$$

$$\vec{r} = \frac{\mu}{m}\overrightarrow{r_{12}}$$

We can repeat the process to find for $\vec{R}$ to get:

$$\vec{R} = -\frac{\mu}{M}\overrightarrow{r_{12}}$$

Extra Note: Sanity check might warn you that your $\vec{R}$ should not have a negative because distance cannot be negative. Recall that $\vec{R}$ is a vector and the negative sign here indicates a flip in *direction*.

[1] An additional benefit of this frame of reference for the two-star system is that it is always centred at a point somewhere in between the two masses, thus reducing the need for complex vector analysis. Thus, from this point on we will use the scale value r and R to denote the magnitude of the vector $\vec{r_S}$ and $\vec{r_E}$ with the origin at the centre of mass for simplicity and r_{12} to denote the distance between sun and Earth (Do note that in general for multiple star system, this simplification is not valid.)

In general, the value of r, R, and r_{12} can be varying with time but the mass of the objects are constant. For the special case of the two-body system, it is always possible to modify the expression for the total energy such that it is equivalent to having one of the star stationary with the other star moving around the stationary star with a mass equivalent to the reduced mass.

(c) **Show that you can always make such a modification to the two-star system by showing that the total energy can be expressed as the following:**

$$E_{total} = \frac{1}{2}\mu\overrightarrow{v_{12}}^2 - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

Extra Info: This is equivalent to the total energy in a coordinate system where one of the stars is stationary and the other star is moving with a speed $|\vec{v_{12}}|$ where $|\vec{v_{12}}|$ is the magnitude rate of change of the distance $\overrightarrow{r_{12}}$. **[3]**

Solution:

From part (b):

$$\vec{r} = \frac{\mu}{m}\overrightarrow{r_{12}}$$

$$\vec{R} = -\frac{\mu}{M}\overrightarrow{r_{12}}$$

We can thus find the corresponding velocity vectors $\vec{v}$ and $\vec{V}$:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{\mu}{m}\frac{d\overrightarrow{r_{12}}}{dt} = \frac{\mu}{m}\overrightarrow{v_{12}}$$

$$\vec{V} = \frac{d\vec{R}}{dt} = -\frac{\mu}{M}\frac{d\overrightarrow{r_{12}}}{dt} = -\frac{\mu}{M}\overrightarrow{v_{12}}$$

With these, we can then substitute back into the expression in **(a)**, together with the fact that $\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}$, to yield the desired result.

(d) **Show that the equation derived in part (c) simplifies to the usual expression for E_{total} when $m << M$. Thus, explain why we can use this simplified expression for the Earth-Sun system as well as any other Sun-(Solar System Planet) pair.** **[2]**

Solution:

When $m << M$, the limiting cases become: $m + M \approx M$ and $\mu \approx m$. Thus the expression in **(c)** becomes:

$$E_{total} = \frac{1}{2}m\vec{v}^2 - \frac{GMm}{|\overrightarrow{r_{12}}|}$$

This is exactly the case for the energy of an object m orbiting a mass M where the larger mass M is fixed in place. Since in most planet to sun mass ratios satisfy $m << M$, it is okay to used the simplified version when analysing the system.

In most orbital questions involving two masses, it is sufficient to further simplify the problem by resolving the motion of the secondary mass into motions in the radial and angular directions as shown in **Figure 2** below.

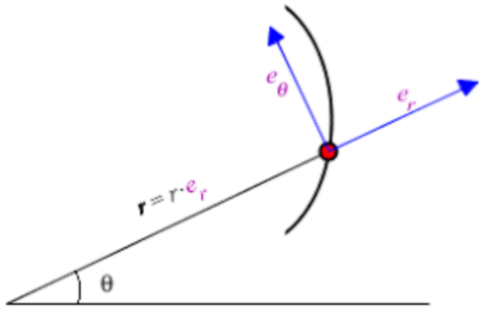

Figure 2: Radial and Angular Components of the Velocity

The radial motion will lead to the secondary mass experiencing periapsis and apoapsis, whereas the angular component together with the gravitational potential adjust according to the radial distance to ensure the total energy is constant, thus maintaining an orbit.

The equation in part (c) can further be simplified by introducing

$$v_{12}^2 = v_\theta^2 + v_r^2$$

Where v_θ^2 and v_r^2 are the angular and radial components of the velocity respectively.

We can define an *effective potential energy* $U_e(\overrightarrow{r_{12}})$ consisting of the angular component of kinetic energy and gravitational potential energy. This will give us the form below.

$$E_{total} - U_e(\overrightarrow{r_{12}}) = \frac{1}{2}\mu v_r^2$$

(e) **Show that the effective potential energy as defined by the above expression is given by:**

$$U_e(\overrightarrow{r_{12}}) = \frac{|\vec{L}|^2}{2\mu|\overrightarrow{r_{12}}|^2} - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

Where $|\vec{L}| = \mu v_\theta |\overrightarrow{r_{12}}|$ is the magnitude of the angular momentum vector, a conserved quantity in this case due to Kepler's Second Law. **[1]**

Solution:

Using the expression given in **(c)** and substitute the expression given for the magnitude of $\overrightarrow{v_{12}}$ to get:

$$E_{total} = \frac{1}{2}\mu(v_\theta^2 + v_r^2) - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

Using the expression given the question for the angular component of the velocity yileds:

$$E_{total} = \frac{1}{2}\mu v_r^2 + \frac{|\vec{L}|^2}{2\mu|\overrightarrow{r_{12}}|^2} - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

From here, we take a look at the expression we are given in the question above this subpart and compare the two. We see that the effective potential energy result follows immediately.

$$E_{total} = \frac{1}{2}\mu v_r^2 + U_e(\overrightarrow{r_{12}})$$

$$E_{total} = \frac{1}{2}\mu v_r^2 + \frac{|\vec{L}|^2}{2\mu|\overrightarrow{r_{12}}|^2} - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

(f) **Sketch this effective potential energy with respect to the distance between the two objects. Using the fact that $E_{total} \geq U_e$ for any orbit, explain using the graph where circular and elliptical orbits occur.** [3]

Solution:

Look at the expression for the Effective Potential:

$$U_e(\overrightarrow{r_{12}}) = \frac{|\vec{L}|^2}{2\mu|\overrightarrow{r_{12}}|^2} - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

Qualitatively we have the following:
We have two terms. The first is in the form $\frac{k_1}{|\overrightarrow{r_{12}}|^2}$ whilst the second is in the form $\frac{k_2}{|\overrightarrow{r_{12}}|}$. To proceed, we see where each term will dominate.

For very small $|\overrightarrow{r_{12}}|$, the inverse quadratic term will dominate. This makes the overall potential positive as well.
For very large $|\overrightarrow{r_{12}}|$, the inverse linear term will dominate. This will make the overall potential negative. It also has the effect of causing the graph to asymptotically approach the x-axis as $|\overrightarrow{r_{12}}|$ tends to infinity.
There will be a point in the middle where the transfer of dominance will take place and the direction of the curve will change. This has the effect of making a "well". Combining it all together we get the desired shape:

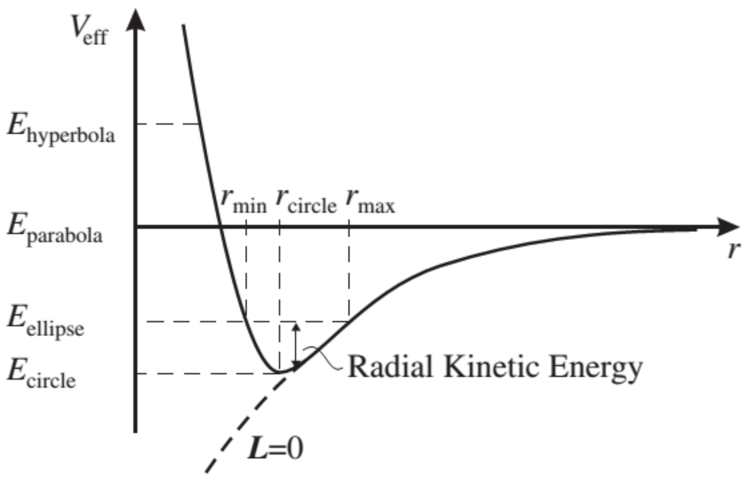

Figure 3: Effective Potential Plot.

Another way to do it is to find the first and second derivatives of the effective potential in $|\overrightarrow{r_{12}}|$ and see the values of each at different $|\overrightarrow{r_{12}}|$ to determine both the gradient and the locations of any extremum/inflection points. You will get the same shape.

Finally, the points for each type of orbit has also been labelled.

Part III Restricted Three-Body Problem

Let us shift gears a little. The energy-analysis approach is massively useful as we have seen earlier. However, let us not discount the force-analysis approach. Sometimes, it gives us a more intuitive feel about the problem and how things will play out.
Even just by using only the force-analysis method, we can theoretically solve for a restricted version of the famous *Three-Body Problem*. In this restricted case, we impose a limit where the mass of the third object must be insignificant to the other two masses. We will not be solving it today in this question but rather study its solutions.

For this restricted problem, the solutions that exist are pretty well known. One famous group is the *Lagrange Points* and there are a total of five of them.

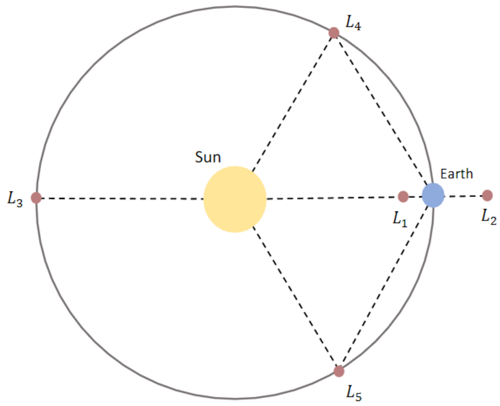

Figure 4: The Five Lagrange points for the Earth-Sun system.

In this case, the location of the points are easier to explain using the idea of balancing forces. They are defined as the points where the gravitational forces from the two large objects and the centrifugal force balance each other. The first three Lagrange points, L_1, L_2, L_3 were in fact discovered by the mathematician Leonhard Euler. It was the final two Lagrange points, L_4 and L_5, which were first discovered by Joseph-Louis Lagrange. That is not to say the energy-approach we did earlier cannot be applied here. The energy-analysis approach is helpful when studying the stability of objects at these Lagrange Points.

(g) **Does the Lagrange Point, L_1, coincide with the Centre of Mass point of the system as per what we defined earlier in part (b)?** **[1]**

Solution:

No.

L_1 is located where the net gravitational attraction from the two masses provides the necessary centripetal acceleration for a third object to remain in the same relative position with the other two. Refer to **Figure 4** and you see that L_1 is closer to Earth. The centre of mass of the Sun-Earth system is closer to the Sun.

The other point of interest in Physics is that of the Centre of Gravity. For our purposes, we shall define this point to be where the gravitational forces by the two objects negate each other.

(h) **Does the Center of Gravity coincide with the Lagrange point L_1?**
Hence, as well as referencing your answer in part (g), roughly sketch the positions of the Center of Gravity, Center of Mass and L_1 for the Earth-Sun System.
Note: The diagram does not need to be to scale but the order of the three points must be shown clearly. **[2]**

Solution:

No.

Figure 5: COM, COG and L_1 points.

The harder part comes from determining the relative positions of L_1 and the COG. Take a look at the force balance at L_1:

$$F_{g,Sun,L1} = F_{g,Earth,L1} + F_{centripetal}$$

At the COG, $F_{g,Sun,COG} = F_{g,Earth,COG}$ by definition. This means that the COG must be closer to Earth than L_1 in order for the Earth's gravitational attraction to equal the Sun's.

The other Lagrange point of interest is that of L_2 due to the launch of the James-Webb Space Telescope (JWST) on the 25 December, 2021. After launch, it made its journey towards L_2.

(j) **Show that the Lagrange Point L_2 is $1.51 \times 10^9 m$ away from Earth.**
Hint: You should use a force analysis here. Additionally, you may find that the resultant equation may not be easily solved analytically. You may instead use any numerical methods to show the desired results instead. **[3]**

Solution:

Following the hint, we start with the force balance for L_2:

$$F_{g,Sun} + F_{g,Earth} = F_{centripetal}$$

$$\frac{GM_\odot m}{d_\odot^2} + \frac{GM_\oplus m}{d_\oplus^2} = m\omega^2 d_{CM}$$

Where m and d_{CM} are the masses of the 3rd object and said object's distance to the Centre of Mass.

Knowing that the system's center of mass is relatively very near to the center of the sun, we will use the approximation that $d_{CM} = d_\odot$. Additionally, we will use a few substitutions: $d_\odot = d_{suntoearth} + d_\oplus = d_{\odot\oplus} + d_\oplus$ and by definition of the Lagrange point: $\omega = \omega_\oplus$.

To our force balance, we will manipulate it to make it:

$$GM_\odot d_\oplus^2 + GM_\oplus d_\odot^2 = \omega^2 d_\odot^3 d_\oplus^2$$

At this juncture, given the nature of the question, one can plug in the values given to show that both sides are equal. Doing this, you get (units omitted):

$$GM_\odot d_\oplus^2 + GM_\oplus d_\odot^2 = 3.116 \times 10^{38}$$

$$\omega^2 d_\odot^3 d_\oplus^2 = 3.119 \times 10^{38}$$

The values are within 0.1%. Given the approximations we made, this is sufficient to show that both sides are equal and the distance to L_2 is $1.51 \times 10^9 m$.

Extra:
For completeness, we will proceed with the analysis. We will make the substitutions to get a monster:

$$G\left[M_\odot d_\oplus^2 + M_\oplus (d_{\odot\oplus} + d_\oplus)^2\right] = \omega_\oplus^2 (d_{\odot\oplus} + d_\oplus)^3 d_\oplus^2$$

$$G\left[M_\odot d_\oplus^2 + M_\oplus d_{\odot\oplus}^2 \left(1 + \frac{d_\oplus}{d_{\odot\oplus}}\right)^2\right] = \omega_\oplus^2 d_{\odot\oplus}^3 \left(1 + \frac{d_\oplus}{d_{\odot\oplus}}\right)^3 d_\oplus^2$$

This equation is not easily solved; it's a quintic in $d_\oplus$. To proceed, we need to simplify it.

We will expect L_2 to be much closer to Earth than to the Sun, so we can approximate that $d_\oplus << d_{\odot\oplus}$. Hence, we can use the Binomial approximation $(1+x)^n \approx 1 + nx$ for small x. Doing so yields:

$$G\left[M_\odot d_\oplus^2 + M_\oplus d_{\odot\oplus}^2 \left(1 + \frac{2d_\oplus}{d_{\odot\oplus}}\right)\right] = \omega_\oplus^2 d_{\odot\oplus}^3 \left(1 + \frac{3d_\oplus}{d_{\odot\oplus}}\right) d_\oplus^2$$

With the approximation, our equation becomes a cubic in $d_\oplus$, which is much easier to solve. Solving it will yield the result:

$$d_\oplus = 1.51 \times 10^9 m$$

Note: Our initial assumption that $d_{CM} = d_\odot$ can be checked. The COM of the Earth-Sun system is displaced from the Sun's centre by $4.48 \times 10^6 m$, which accounts for a 0.0003% difference.

(k) **What are the angular diameters of the Sun and Earth at L_2?** **[2]**

Solution:

From geometry, we can get an object's angular diameter:

$$\theta = 2\tan^{-1}\left(\frac{R}{d}\right)$$

Where θ, R and d are the object's angular diameter, radius and distance respectively. We can prove later that the Sun and Earth are tiny at L_2, so the small angle approximation is allowed too:

$$\theta = 2\left(\frac{R}{d}\right)$$

Thus:

$$\theta_{\oplus} = 2\left(\frac{R_{\oplus}}{d_{\oplus}}\right) = 8.437 \times 10^{-3}\text{rad} = 0.483°$$

$$\theta_{\odot} = 2\left(\frac{R_{\odot}}{d_{\odot\oplus}}\right) = 9.215 \times 10^{-3}\text{rad} = 0.528°$$

(m) **State one reason why L_2 was chosen as the point to park the JWST?** **[1]**

Solution:

Because relative to L_2, the Sun, Earth, and Moon are all roughly in the same direction, and so the sun-shield, when deployed, can block off all 3 major infrared sources simultaneously.

From QM: The point of part (k) was to show that the Earth is a terrible sun-shield and subtly hint participants to the right answer here.

Question 3 A Peek at Infinity

Spacetime

The field of Cosmology studies the universe at the largest scale. One part of Cosmology is understanding the evolution of the universe. It uses a few concepts from both Special and General Relativity. One of such concepts is about the idea of Spacetime, which you can think of as the stage for which all events in the universe perform on.

Thanks to the Astronomer Edwin Hubble, we know today that this "stage" is expanding. Understanding the nature of this expansion is thus key to understanding the plausible futures of the universe as well as piece together its history; these we shall soon discover.

Part I Expansion

Let us first consider a “1+1” Minkowski Spacetime, a spacetime that obeys the *Lorentz* Transformation. In this case, our Minkowski Spacetime will be 1 dimension of flat[2] Space - on the x-axis, with 1 dimension of time on the y-axis. We will use the coordinate system below to represent our spacetime.

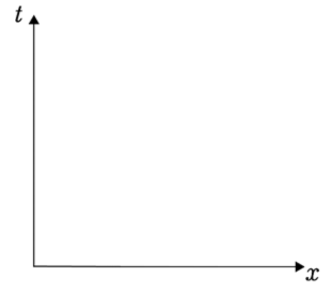

Figure 6: A "1+1" Minkowski Spacetime Diagram

On that diagram, we can label an event which occurs at a *particular location* in space at a *particular time*. This "event" is not limited to abstract ideas. It can also refer to any physical object at a particular location in space at a particular time.

Let us extend this idea. If we plot an event at *multiple* locations and time points, we get a series of dots. By doing this continuously, all the dots get smushed together until we get a line. This line thus represents the trajectory of the event through space *and* time. Remember, the coordinates of a point on this diagram gives you the location and time the plotted event takes place.
For example, the diagram below depicts two **stationary** objects at $x = x_1$ and $x = x_2$ respectively.

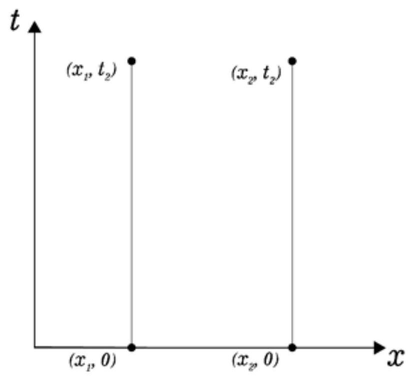

Figure 7: 2 stationary objects on a Minkowski Spacetime Diagram

[2]Euclidean

For a static universe, space does not expand. In such a non-expanding scenario, the infinitesimal (very small) distance between two points in this coordinate system is given by the following:

$$ds^2 = -c^2dt^2 + dx^2$$

Where c is the speed of light in a vacuum, dt is the infinitesimal (very small) change in time and dx is the infinitesimal (very small) change in the x-coordinate between two points. The following figure illustrates it:

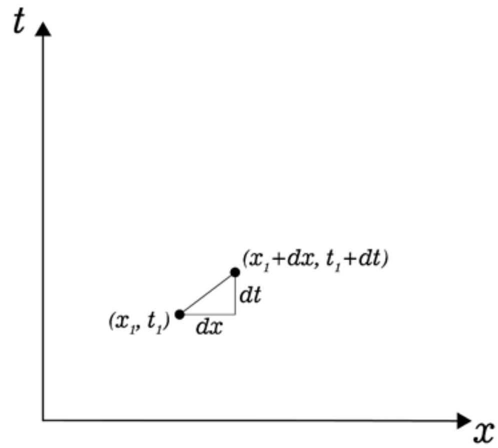

Figure 8: Spacetime interval on a "1+1" static Minkowski Spacetime Diagram

(a) **Taking the case of the non-expanding Minkowski Space, plot the trajectory of an object that moves in the positive x-direction by b units with a velocity of b/t and then immediately moves back to its original location with the opposite velocity. Label your diagram clearly.** [1]

Solution:

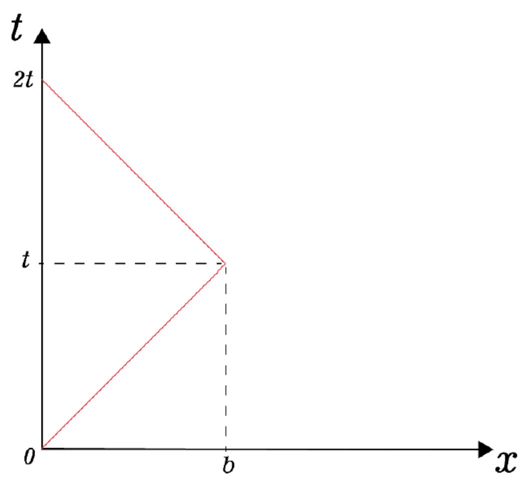

Figure 9: Minkowski Diagram for a uniform speed object that reflects back at $x = b$.

That was for the case where space was not expanding. In the case of an expanding space[3] however, the infinitesimal distance between two points is now given by:

$$ds^2 = -c^2dt^2 + a^2(t)dx^2$$

This extra $a(t)$ term is what we call the **Scale Factor**. In this case, it is the scale factor of the x-axis and is how we mathematically factor expansion into our spatial dimension.

For two objects pinned to $x = x_1$ and $x = x_2$, when $a(t) = 2$, their physical separation in space has become twice that of when $a(t) = 1$ even though their x-coordinates in our spacetime diagram *did not change*. This is exactly the case we see in the real world, where objects do not move[4] but rather the space between them expands which increases their physical separation.

To expand on this point, a common analogy used to illustrate this point are two marker dots on the surface of an inflating balloon. These dots do not move, but because the balloon is expanding, the two marker dots gets pushed apart.

The scale factor is a *function of time* t. For an expanding universe, that means that $a(t)$ increases with time t. This implies that the distance between two stationary objects with constant x-coordinates in our spacetime diagram will be increasing at $\frac{da}{dt}\Delta x$ where Δx is the distance along the diagram's x-axis between the two objects.

(b) **Take the case of a *constantly* expanding space. Draw the path that a light pulse takes, as it is emitted from a stationary observer at position $x = x_1$ and time $t = 0$, to a stationary observer at $x = x_2$**

Hint: In a short time interval dt, light travels $a(t)dx$. Will this correspond to a straight line in the diagram? **[15]**

Solution:

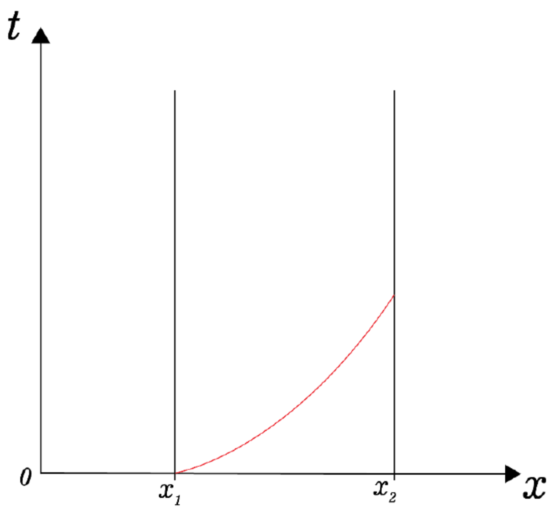

Figure 10: Minkowski Diagram for a uniform velocity object in expanding space.

Note that the exact curve (Quadratic, Cubic, Exponential, etc.) is not required, just that the gradient of your worldline needs to increase.

[3] We shall limit to expansion only in the x-axis.

[4] Because of relativity and the dependency on a reference frame for velocity, you cannot say something "does not move" without also giving the point of reference. But we can still make a physical distinction between movement through Space and the movement due to the expansion of Space. This caveat shall be ignored for our purposes and left as a topic to be discussed on another day.

This first light pulse that was sent has 10 cycles. Next, an identical light pulse is emitted again from $x = x_1$, just as the tenth cycle of the first light pulse ends at time $t = t_1$.

(c) **On the diagram you have drawn, draw the path that the second light pulse took and compare the time interval between receiving the first and second pulses at $x = x_2$ with t_1.** [15]

Solution:

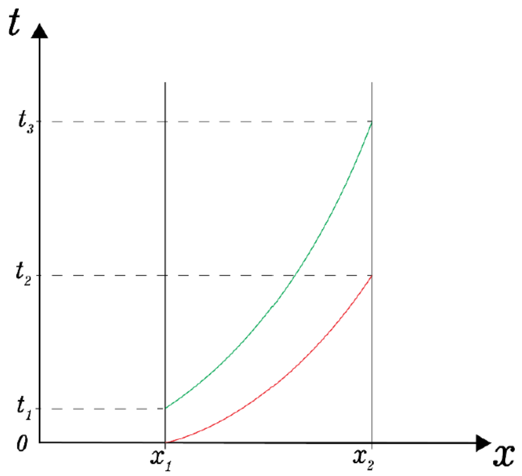

Figure 11: Two light beams in expanding space.

The time interval $t_3 - t_2 > t_1$, the gradient of the two light paths should be the same at the *same time* (horizontally) and increasing.

Note: $t_1 \neq t_2$

(d) **Following your answer in part (c), what does this imply about the wavelength of light received at x_2 compared to when it was emitted at x_1?** [15]

Solution:

Since the time interval to receive 10 cycles of the light pulse is longer than that to emit them, and light travels at a constant speed c in the vacuum of space, the length of the light pulse must thus have increased and the wavelength of the light has increased.

In Cosmology, *Hubble's Law* defines the relationship between the apparent recession velocity, due to expansion, and proper distance (distance between two points in space at the same cosmological time).

(e) **Show that in our "1+1" Minkowski Spacetime, Hubble's Law is given by the following:**

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Hint: The recession velocity is the time differential of proper distance. The proper distance between two points in space at the same cosmological time is given by ds when $dt = 0$. **[2]**

Solution:

We start with Hubble's Law and replace with the necessary variable:

$$\begin{aligned} v &= H(t)d \\ \frac{ds}{dt} &= H(t)d \\ &= H(t)a(t)\Delta x \\ &= \frac{d}{dt}(a(t)\Delta x) \end{aligned}$$

Using Product Rule:

$$\begin{aligned} \frac{ds}{dt} &= \frac{d}{dt}(a(t)\Delta x) \\ &= \frac{da(t)}{dt}\Delta x + a(t)\frac{d(\Delta x)}{dt} \\ &= \dot{a}(t)\Delta x + 0 \\ &= H(t)d \end{aligned}$$

Thus:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

In cosmology, one of the ways we can define the size of the Observable Universe is the maximum distance from where light has had enough time to travel and reach us. To do this, we first must find the age of the Universe.
The age of the Universe as a function of scale factor is given by the following:

$$\Delta t = \int_0^{t(a)} dt$$

Where $t(a)$ is the age of the universe at a particular scale factor a.

From before, we know that the scale factor changes with time. For the sake of our understanding today, we will model the change of the scale factor according to the formula below:

$$\frac{da}{dt} = \frac{\kappa}{a}$$

where κ is a positive constant.

(f) **Using the given information, find an expression for the age of the universe as a function of the scale factor.**
Hint: You can treat da/dt as a fraction and change dt to be the subject of the equation. When doing so, you will need to perform a change in limits from $t(a)$ to $a(t)$ when you perform a change in variable for the integration. **[15]**

Solution:

We do a change in variable:

$$dt = \frac{a}{\kappa} da$$

$$\int_0^{\Delta t} dt = \int_0^{a(t)} \frac{a}{\kappa} da$$

$$= \frac{a^2(t)}{2\kappa}$$

(g) **From your answer in part (f), conclude whether the model used for the rate of change in the scale factor is consistent with what is happening in our current universe? Explain your conclusion.** **[1]**

Solution:

No. The rate of expansion of the universe is currently increasing instead of decreasing. In fact, the rate of the expansion of the universe seems to be converging asymptotically to $\dot{a} = \kappa a$ instead.

Part II Density

So far what we have done in the previous section is look at **how** the expansion of space can be modelled. This leaves an unsolved question: What gave rise to this expansion?

Our current understanding of the Universe makes use of the Big Bang Model, when space and time came into existence and space has been expanding ever since. In that model, the rate of expansion is governed by the various densities throughout the universe's history.

(h) **While the size of our observable universe is finite, the size of the actual universe itself may well be infinite.**
Hence, explain what is meant when we say the Big Bang started off from a singularity even though it is impossible for a single point to expand into infinity, given the finite history of the Universe. **[1]**

Solution:

What started from a singularity is the size of the Observable Universe, since light needed time to reach us, the size of the Observable Universe when the age of the Universe is 0 will be 0. The size of the Observable Universe then expanded as the Universe ages.

Extra Notes: If the Universe is infinite (which would be the case if it is flat and there is no centre of expansion), it would have been infinite at the moment of its creation – the Big Bang. Since space came into existence at the Big Bang, the Big Bang happened everywhere in the Universe. From our perspective, everything is moving away from us, at least whatever we can see in our Observable Universe. Every point in space has their own finite Observable Universes - the bubble of light that had time to reach them since the birth of the universe. These bubbles would have zero size at the moment of the Big Bang.
For further reading: one can visit https://physics.stackexchange.com/questions/136860/did-the-big-bang-happen-at-a-point

As mentioned at the start of this question, this field borrows concepts and understanding from Relativity. From Einstein's General Relativity, mathematician Alexander Friedmann derived equations governing the expansion of the universe. Today we know them as the Friedmann[5] Equations. Equation (1) below is one of the results that Friedmann derived, which governs the rate of expansion of space.

$$\frac{d^2R}{dt^2} = \left(-\frac{4}{3}\pi G\left[\rho_m + \rho_{rad} + \rho_\Lambda + \frac{3(P_m + P_{rad} + P_\Lambda)}{c^2}\right]\right) \tag{1}$$

Where:
1. R is the scale factor of the universe, analogous to the scale factor a(t) of the x-axis as we have discussed before in the previous section.
2. G is the Universal Gravitational Constant.
3. ρ_m, ρ_{rad}, and ρ_Λ are the equivalent mass density of matter, radiation, and cosmological constant respectively.
4. P_m, P_{rad}, and P_Λ are the pressure exerted by the matter, radiation, and cosmological constant, on space respectively.
5. c is the speed of light in a vacuum.
6. Λ is the cosmological constant.
7. k is a constant that indicates the curvature of space. When $k > 0$, space has positive curvature; when $k = 0$, space has no curvature; and when $k < 0$, space has negative curvature.

In short, equation **(1)** denotes how the acceleration of the scale factor of the universe is affected by the presence of matter, radiation, and the cosmological constant. Matter and Radiation were entirely created from the moment of the Big Bang. The cosmological constant, however, is a property of free space itself. Thus, it is constantly being produced through the expansion of the universe as expansion results in more space.

[5]They are actually also attributed to 3 other people who arrived at them independently: Georges Lemaître, Howard P. Robertson and Arthur Geoffrey Walker. For convenience we shall just refer to them as the "Friedmann equations".

Now you may be thinking – why would radiation have mass? Aren't photons massless? You would be right, but light carries energy, and with Einstein's famous mass-energy equivalence in the form of $E = mc^2$, we can assign a "mass" to the energy of radiation. We call that the *relativistic mass* of the photons responsible for radiation. This is a parallel to the rest mass that forms most of the mass of matter.

Friedmann had two equations. We have seen the first, now we shall introduce his other equation which outlines the relationship between the equivalent mass density and pressure. This relates the change in equivalent mass (and hence energy) densities as the volume of space increases to the work done by the pressures the various components of our universe exert.

$$\frac{d\rho}{dR} = -3\left(\frac{\frac{P}{c^2} + \rho}{R}\right) \tag{2}$$

(j) **Given that $\rho_m \propto R^{-3}$, $\rho_{rad} \propto R^{-4}$ and $\rho_\Lambda \propto R^0$, show the following:**

$$P_m = 0$$

$$P_{rad}(\rho_{rad}) = \frac{1}{3}\rho c^2$$

$$P_\Lambda(\rho_\Lambda) = -\rho c^2$$

[3]

Solution:

The key here is to transform the proportionality to equality by adding a proportionality constant. We do for matter first.

$$\frac{d}{dR}(k_m R^{-3}) = -3\frac{P/c^2 + \rho}{R}$$

$$-3k_m R^{-4} = -3\frac{P/c^2 + \rho}{R}$$

Isolating for Pressure:

$$P = (k_m R^{-3} - \rho)c^2$$

$$= 0c^2 = 0$$

Repeating these same steps for Radiation and Cosmological Constant will yield the other two results.

(k) **Hence, with reference to equation (1), suggest how the presence of matter, radiation, and the cosmological constant influence the rate of expansion of space.** **[3]**

Solution:

The combined effect from matter is:

$$\rho_m + 3 \times 0/c^2 = \rho_m$$

The combined effect from radiation is:

$$\rho_{rad} + 3 \times \left(\frac{1}{3}\rho_{rad}c^2\right)/c^2 = 2\rho_{rad}$$

From the cosmological constant:

$$\rho_\Lambda + 3 \times \left(-\rho_\Lambda c^2\right)/c^2 = -2\rho_\Lambda$$

Hence, matter and radiation make the acceleration of the expansion of the universe more negative and slows down the expansion of space while the cosmological constant makes the acceleration of the expansion of the universe more positive and speeds up the expansion of space.

(m) **With reference to how ρ_m, ρ_{rad}, and ρ_Λ changes with R, arrange the periods when the expansion of the universe was dominated by the influence of radiation, matter, and the cosmological constant in chronological order.** [2]

Solution:

Since the equivalent mass density of radiation decreases by the fourth power of scale factor while that of matter only decreases by the third power and as R approaches zero, the finite value of the equivalent mass density of the cosmological constant must be smaller than those of radiation and mass which tends towards infinity; the equivalent mass density of radiation will be the highest at the start of the universe and it dominates in influencing the expansion of the universe.

As the equivalent mass density of radiation decreases more rapidly than that of matter, matter dominated period of the universe's expansion comes next.

Finally, as both the equivalent mass densities of radiation and matter decrease as space expands, the cosmological constant, whose equivalent mass density remains constant with expansion, dominates the expansion of the universe, as is the case today.

(n) **Why can't the expansion of our current universe be explained by a universe filled with matter and radiation alone?** [1]

Solution:

As the Supernova Cosmology Project and the High-Z Supernova Search Team showed in 1998, the universe is expanding at an accelerating pace. With only matter and radiation, the equivalent mass density of pressure of radiation and matter only serves to slow down the expansion, and not speed it up, as shown in Equation 2.

Note: Just stating that the universe is expanding at an accelerating pace is sufficient.

Extra Notes for Question 3

This question has brought to light some topics from General and Special Relativity as well as Cosmology. Given the broadness and difficulty of this topic, we recommend some extra texts for further reading:

1. For an introduction to Cosmology without requiring any GR knowledge, the classic text is that by Ryden (2002).
2. For SR, some recommended texts are: Taylor and Wheeler (1966), Mermin (1989), Arzelies (1966) and more.
3. For introductory GR and for students who are more comfortable with the ideas of SR can consult Schutz (2009) and Hartle (2002) for a brief mathematical discussion of it before progressing to GR. Another text would be the one by Misner, Wheeler and Thorne (1973).

Question 4 Lights, Camera, Chill

Astrophotography

As the prices of DSLRs drop since their inception, more people now can take photographs of the night sky. The art of Astrophotography has caught on to become a serious pastime for amateur astronomers.

Part I Photography Jargon

Before we jump in, we should familiarise a little on some technical jargon that is commonly used in the photography community. The ultimate goal of a photographer is to control the amount of light captured by the sensor when taking a photo. It cannot be too much light else the photo will come out overexposed (too bright). The converse is true too, when the camera does not capture enough light.

To achieve a good control over light, photographers concern themselves with 3 main settings:

Camera Settings	**What it Controls**
Shutter Speed / Exposure Length	The length of time the sensor is exposed to light. (e.g. 1/3s, 1/1000s, 1s, etc.)
ISO (pronounced "eye-soh")	The Sensitivity of the sensor. Higher ISO = More Sensitive
f-Number/Ratio	Aperture size of the lens. Same definition with a telescope's f-ratio.

Table 1: The 3 main camera light control parameters.

These three settings are what a photographer uses to control the amount of light that hits the sensor.

The last setting is known as the Zoom. It controls how magnified the image will be. It is usually expressed as the focal length of the lens. (e.g., 18mm, 35mm, 120mm etc.)

Part II Camera Properties

James has recently gotten himself an entry-level DSLR camera and wants to try his hands at Astrophotography. He is planning a road trip to a nearby dark-sky site to test what he can possibly capture. The table below shows some basic specifications of the camera he is using:

Camera Property	Value
Model No.	Nikon D3500
Sensor Size (Width × Height)	23.5mm × 15.6mm
Sensor Pixel Resolution (Width × Height)	6000px × 4000px

Table 2: Basic Specifications of James' Camera

James wants to plan his shot, so he decides to understand a bit more of how his camera will work before leaving. He starts by analysing the camera itself.

(a) **If we assume that the pixels in his sensor are squares, calculate the size of each pixel in micrometers. Give your answer to 1 d.p.** [1]

Solution:

We can pick to calculate either by width or height:

1. Width:

$$l = \frac{23.5}{6000} = 3.9\mu m$$

2. Height:

$$l = \frac{15.6}{4000} = 3.9\mu m$$

Either way we get the same answer to 1 d.p.

Thinking about it a little more, when James connects his camera lens to the camera, the setup essentially functions like a telescope where the focal length of the scope is equals to that of the lens' focal length.

He wants to know how much of the sky he can take given a certain lens focal length and his sensor size. In essence, he wants to know how big a heavenly body with a certain angular diameter θ will appear on his sensor and thereafter the image taken.

In technical terms, he wants to find the *Plate Scale* of his setup.

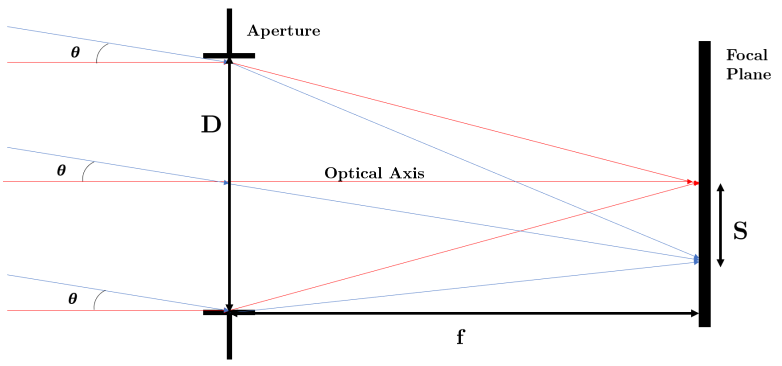

Figure 12: Plate Scale of a Telescope

(b) **Given an optical setup with a focal length f, show that the Plate Scale (PS), in units of arcseconds/mm, of the setup is given by**

$$PS = \frac{206265}{f}$$

[2]

Solution:

Every object with an angular size θ in the sky will be focused into an image of size S based on the optical set-up.
We refer to the figure and get the desired geometry. We focus on the ray that passes through the principal axis of the lens.

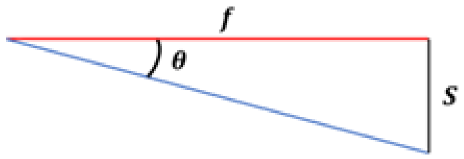

Figure 13: Plate Scale Geometry

We will get the desired:

$$\tan\theta = \frac{S}{f} \approx \theta$$

Where θ is the angular diameter of the object in **radians**, S is the image height and f is the focal length.
To convert radians to arcseconds as required:

$$1 \text{ radian} = \frac{180}{\pi}^{\circ} \approx 206265"$$

Since the question wants the Plate Scale, we find the image size of a hypothetical $\theta = 1"$ angular diameter object:

$$\frac{S}{f} = \frac{1}{206265}$$

$$PS = \frac{\theta}{S} = \frac{206265}{f}$$

In this question, for simplicity, we shall treat James' camera setup as something like that of a Newtonian telescope. In reality, the camera lens will change the optics slightly.

The important thing to note in this question is that the lens will act as the Optical Tube Assembly (OTA) of a telescope setup. The lens' focal length will be analogous to that of the OTA's focal length.

(c) **Given that James hooks a camera lens with a focal length of 300.0mm, calculate the Plate Scale of his camera setup.** [1]

Solution:

We use the formula provided in **(b)**.

$$PS = \frac{206265}{f} = 687.6"/mm$$

(d) **If James wants to have the full moon fit just nicely in his image, what focal length does he need?** [2]

Solution:

The angular diamter of the Moon is given by:

$$\theta_{Moon} = \approx 2 \times \frac{R_{Moon}}{d_{Moon}}$$
$$= 0.518°$$

Note: A small-angle approximation was used here. Using arctan would yield the same result to 3 d.p.

We want the moon to fill the image. Since the image is rectangular, this means that the limiting side will be the image's height. So we want the moon to span the height of the image.

From **(a)**, the height of the sensor is 15.6mm. For the moon to span the height in the image, we find that the required PS is:

$$PS_{req} = \frac{\theta_{moon}}{h_{sensor}} = 120"/mm$$

Using the relation given in **(b)**, we find that the required focal length is:

$$f = \frac{206265}{PS_{req}}$$
$$= 1720mm$$

Note: This value may seem big but realistically we usually put the camera sensor to the back of a telescope. A C6 with a focal length of 1500mm usually fills up the whole image with the moon. Our answer isn't too far off.
Differences in values will be because of the assumptions we made in this question.

Part III Planning the Shot

Because of the low-light level nature of astrophotography, photographers will have to take long-exposures. This works by allowing more photons to hit each pixel on the sensor, the more photons that hit a pixel, the brighter the output from that pixel will be (brighter pixel).

Note that it does not time average the brightness. The brightness of a pixel purely represents the number of photons that hit that pixel when said pixel was exposed to light.

Because of this and the fact that the sky rotates as time passes, a star or any source of light will start leaving trails in the image if the camera is not rotating along with the sky. This is how we get star-trail photos.

James has this sweet zoom lens with a focal length of 300.0mm. He intends to take the photo of the **M11**, the Wild Duck Cluster. It is a rather bright star cluster close to the Celestial Equator. Because James just got into photography, he does not have a tracking mount for his camera.

(e) **James wants to avoid having star trails in his image of M11. What is the maximum amount of time he can leave his sensor exposed before ALL the stars start visibly appearing as star trails in his image? You can treat the stars as point sources in this question.**

Note: Star trails start becoming visibly apparent when a source that covers 1px stretches out to span 5px in any direction. **[4]**

Solution:

Star trails form because the camera is not tracking the stars as they move across the night sky. To see when this becomes visible, we need to find out what is the maximum distance these stars can travel become crossing the threshold.

We start by first checking that the note given in the question applies. Because the stars are treated as point sources, their image will comfortably be contained within 1px of the sensor. Thus, the next step is finding how long it takes for the image of a star to traverse 5 pixels.

The longest distance contained in a square is its **diagonal**.

Thus, the distance this point star's image needs to move across the sensor is hence:

$$s = 5l_{px}\sqrt{2} = 27.6nm$$

Because this setup is the same as part **(c)**, the $PS = 687.6\text{"}/mm$. Thus, using these two pieces of information we can find out how much the stars need to travel **across the sky** to produce an image that travels $27.6nm$ on the sensor:

$$\theta = PS \times s = 18.98\text{"}$$

We have this distance, we just need the speed at which the stars travel to find time. Since we are photographing the Wild Duck Cluster (M11) that is located near the Celestial Equator (CE) and having a rather zoomed in view, we can assume that the stars imaged all move across the night sky at the same rate as those stars on the CE. Thus, we can then find the answer:

$$T = \frac{\theta}{\omega} = \frac{18.98\text{"}}{15°/h}$$

$$= 1.27s$$

Note from QM: This question, although engineered slightly, was meant to give an introduction to the 300/500 rule in Astrophotography. The rule-of-thumb is a handy way of calculating the longest exposure time I can take before star trails become apparent. To use it, simply do the following:

$$T_{max} = \begin{cases} \frac{300}{f} & \text{For crop-sensors} \\ \frac{500}{f} & \text{For full-frame sensors} \end{cases}$$

Additional Note: M11 has a declination of $\delta = 06°16'12"$. If we were to centre our camera on it and positioned the camera in such a way that we maximised the range of declination captured using the longer edge, we will be able to observe stars with declination of $\delta_m = 08°30'51.3"$. Their angular speed would be $\omega_m = \omega \cos \delta_m = 14.83°/h$. This constitutes a 1% difference.

(f) **If James switches to his 35.0mm lens to get a wide-field shot of the winter night sky, give a brief explanation on whether his maximum exposure time will go up or down.** **[2]**

Solution:

His maximum exposure time will go up.

Having a wider field, the images of the stars will travel slower across the sensor. Thus, to cover the same distance, a longer amount of time is needed and thus our maximum exposure time before star trails become visible increases.

Note: This question is not asking about overexposure as I did not give you data on the other settings of the camera like ISO, f/number etc. It is a real concern though.

Part IV Reality and Noise

In the real world and because of how a sensor works, sometimes a sensor pixel will randomly send an output signal saying it got hit by a photon when it did not. This is the source of *Noise* in a picture and gives rise to a grainy background.

In Astrophotography, noise is much more apparent as we work in a low-light environment and noise can become significant. There are multiple sources where such noise can come from, but today we will only limit ourselves to that of random thermal noise.

We will define the quality of an Astro image using the concept of Signal-to-Noise Ratio (SNR), where the signal (stars) is much brighter than the noise. A higher SNR is a better image in our books. Mathematically, we can define the SNR of an image/group of pixels as follows:

$$SNR = \frac{\text{Total Source Count}}{\sigma_{Noise}}$$

$$\sigma_{Noise} = \sqrt{N_{px} \cdot r \cdot T}$$

Where N_{px} is the number of pixels of interest (how many pixels the object covers on the sensor), T is the exposure length and r is the thermal noise rate of the sensor of **3 counts/px every minute**.

After reading up about noise, James consults his Astronomy teacher who did his thesis on radio and CCD imaging. There, he is introduced the idea of a *Zero-Point Magnitude* (ZPM). In short, it is the measure of the sensitivity of the sensor. Recalling more, James remembered his teacher saying:

"If I say my sensor's ZPM is 21, it means that when I expose my whole sensor to a source of apparent magnitude 21 and allow all the pixels to see it, each and every pixel in the sensor will output 1 count per second.
So when I expose the sensor to a brighter source, the output count rate will increase proportionally with the intensity of the source."

(g) **If James' camera sensor has a ZPM of 16 mag, calculate the SNR for a point-source star of apparent magnitude 10, given that James did a 3-second exposure with his 120mm lens.** [4]

Solution:

We first need to find the Count Rate (CR) for a star with magnitude 10. Since the output CR is dependent on the number of photons hitting it (i.e. Light Source's Intensity): $CR \propto I$.

We can use Pogson's Law and replace I with CR and other relevant variables.

$$m_2 - m_1 = -2.5 \lg \left(\frac{I_2}{I_1} \right)$$

$$m_{star} - m_{ZPM} = -2.5 \lg \left(\frac{CR_{star}}{CR_{ZPM}} \right)$$

Since we want to find the CR for a magnitude 10 star:

$$CR_{star} = CR_{ZPM} \times 10^{\frac{m_{ZPM} - m_{star}}{2.5}}$$
$$= 251.2 \text{ count/s}$$

Again, the star will fit into 1 pixel, thus:

$$N_{px} = 1$$

Thus, the noise is correspondingly:

$$\sigma_{noise} = \sqrt{N_{px} \cdot r \cdot T} = \sqrt{1 \cdot \left(\frac{3}{60}\right) \cdot 3}$$
$$= \frac{\sqrt{15}}{10}$$

Thus, the SNR is:

$$SNR = \frac{\text{Total Source Count}}{\sigma_{noise}} = \frac{CR_{star} \cdot T}{\sigma_{noise}}$$
$$= 649$$

Note from QM: In radio and other forms of observational astronomy, our objects have a finite size and is also limited by Seeing. Therefore, the image will span over multiple pixels and this is where N_{px} comes in. There are also other forms and sources of noise that are affected by the equipment and environment. The idea still stands but the two equations given will change slightly.
For further reading, see Readout Noise and Dark Current.

Part V Targeting

James has decided to go on his adventure on 30 June 2022. His goal is to capture the bulge of the Milky Way which passes through the constellation of Sagittarius. That night is the night of a New Moon.

The star *Alnasl* (γ Sgr) in the teapot asterism is almost right smack in the middle of the bulge of the Milky Way.

Alnasl Stellar Details	Value
Right Ascension	$18^h07^m16^s$
Declination	$-30°25'23.9''$

Table 3: Alnasl Stellar Details

(h) **Given further details that James will be at $2.540°N, 103.82°E$, estimate the local civil time when *Alnasl* reaches upper culmination. Local Noon occurs at 13:08 local civil time.**

Note: The location also follows the time zone GMT+8, identical to Singapore. **[4]**

Solution:

Alnasl will reach upper culmination when it crosses the Meridian. This happens when the Local Sidereal Time (LST) equals to the RA of *Alnasl*.

The date is 30 June, which is 9 days after the Summer Solstice. This means the RA of the Sun is:

$$RA_{\odot} = 6^h + \frac{9}{365.25} \times 24^h = 06^h35^m29^s$$

Since local noon was at 1308hrs local civil time, this means that the LST at that Local Civil Time (LT) was $06^h35^m29^s$.

Thus, the LT when *Alnasl* reaches Upper Culmination is:

$$\begin{aligned} LT &= LST + \Delta T = LST + (LT_{Noon} - LST_{Noon}) \\ &= 18^h07^m16^s + (13^h08^m - 06^h35^m29^s) \\ &= 24^h39^m47^s \\ &= 00^h39^m47^s \end{aligned}$$

Thus *Alnasl* will reach upper culmination at LT 0039hrs.

~ **FIN** ~